HITCHHIKER'S GUIDE
TO THE FRACTIONAL SOBOLEV SPACES

ELEONORA DI NEZZA, GIAMPIERO PALATUCCI, AND ENRICO VALDINOCI

CONTENTS

1. INTRODUCTION

These pages are for students and young researchers of all ages who may like to hitchhike their way from 1 to $s \in (0,1)$. To wit, for anybody who, only endowed with some basic undergraduate analysis course (and knowing `where his towel is`), would like to pick up some quick, crash and essentially self-contained information on the fractional Sobolev spaces $W^{s,p}$.

The reasons for such a hitchhiker to start this adventurous trip might be of different kind: (s)he could be driven by mathematical curiosity, or could be tempted by the many applications that fractional calculus seems to have recently experienced. In a sense, fractional Sobolev spaces have been a classical topic in functional and harmonic analysis all along, and some important books, such as [53, 80] treat the topic in detail. On the other hand, fractional spaces, and the corresponding nonlocal equations, are now experiencing impressive applications in different subjects, such as, among others, the thin obstacle problem [77, 61], optimization [35], finance [24], phase transitions [2, 12, 78, 36, 41], stratified materials [73, 21, 22],

anomalous diffusion [60], crystal dislocation [82, 43], soft thin films [51], semiperme-able membranes and flame propagation [13], conservation laws [8], ultra-relativistic limits of quantum mechanics [37], quasi-geostrophic flows [58, 25, 19], multiple scattering [34, 23, 46], minimal surfaces [14, 18], materials science [4], water waves [71, 89, 88, 30, 27, 65, 31, 32, 29, 28, 38, 47, 66, 33] and elliptic prob-lems with measure data [62, 63, 49]. `Don't panic`, instead, see also [76, 77] for further motivation.

For these reasons, we thought that it could be of some interest to write down these notes – or, more frankly, we wrote them just because `if you really want to understand something, the best way is to try and explain it to someone else.`

Some words may be needed to clarify the style of these pages have been gathered. We made the effort of making a rigorous exposition, starting from scratch, trying to use the least amount of technology and with the simplest, low-profile language we could use – since `capital letters were always the best way of dealing with things you didn't have a good answer to.`

Differently from many other references, we make no use of Besov spaces[1] or in-terpolation techniques, in order to make the arguments as elementary as possible and the exposition suitable for everybody, since `when you are a student or what-ever, and you can't afford a car, or a plane fare, or even a train fare, all you can do is hope that someone will stop and pick you up`, and `it's nice to think that one could, even here and now, be whisked away just by hitchhiking.`

Of course, by dropping fine technologies and powerful tools, will miss several very important features, and we apologize for this. So, we highly recommend all the excellent, classical books on the topic, such as [53, 80, 1, 83, 84, 90, 70, 81, 59, 54], and the many references given therein. Without them, our reader would remain just a hitchhiker, losing the opportunity of performing the next crucial step towards a full mastering of the subject and becoming the captain of a spaceship.

In fact, compared to other Guides, this one is not `definitive`, and it `is a very evenly edited book and contains many passages that simply seemed to its editors a good idea at the time.` In any case, of course, we know that we cannot `solve any major problems just with potatoes` – it's fun to try and see how far one can get though.

In this sense, while most of the results we present here are probably well known to the experts, we believe that the exposition is somewhat original.

---

[1]About this, we would like to quote [48], according to which "The paradox of Besov spaces is that the very thing that makes them so successful also makes them very difficult to present and to learn".

These are the topics we cover. In Section 2, we define the fractional Sobolev spaces $W^{s,p}$ via the Gagliardo approach and we investigate some of their basic properties. Then, in Sections 3 and 4 we focus on the Hilbert case $p = 2$, dealing with its relation with the fractional Laplacian, and letting the principal value integral definition interplay with the definition in the Fourier space, as well as with the one that uses the Riesz potential.

Section 5 is devoted to the extension problem of a function in $W^{s,p}(\Omega)$ to $W^{s,p}(\mathbb{R}^n)$: technically, this is slightly more complicated than the classical analogue for integer Sobolev spaces, since the extension interacts with the values taken by the function in Ω via the Gagliardo norm and the computations have to take care of it.

Sobolev inequalities and continuous embeddings are dealt with in Section 6, while Section 7 is devoted to compact embeddings. Then, in Section 8, we point out that functions in $W^{s,p}$ are continuous when sp is large enough.

In Section 9, we present some counterexamples in non-Lipschitz domains.

After that, we hope that our hitchhiker reader has enjoyed his trip from the integer Sobolev spaces to the fractional ones, with the `advantages of being able to get more quickly from one place to another – particularly when the place you arrived at had probably become, as a result of this, very similar to the place you had left.`

The above sentences written in `old-fashioned fonts` are Douglas Adams's of course, and we took the latitude of adapting their meanings to our purposes. The rest of these pages are written in a more conventional, may be boring, but hopefully rigorous, style.

2. The fractional Sobolev space $W^{s,p}$

This section is devoted to the definition of the fractional Sobolev spaces.

No prerequisite is needed. We just recall the definition of the Fourier transform of a distribution. First, consider the Schwartz space $\mathscr{S}$ of rapidly decaying C^∞ functions in $\mathbb{R}^n$. The topology of this space is generated by the seminorms

$$p_N(\varphi) = \sup_{x \in \mathbb{R}^n} (1 + |x|)^N \sum_{|\alpha| \le N} |D^\alpha \varphi(x)|, \quad N = 0, 1, 2, \dots,$$

where $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Let $\mathscr{S}'(\mathbb{R}^n)$ be the set of all tempered distributions, that is the topological dual of $\mathscr{S}(\mathbb{R}^n)$. As usual, for any $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we denote by

$$\mathscr{F}\varphi(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \varphi(x) \, dx$$

the Fourier transform of φ and we recall that one can extend $\mathscr{F}$ from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$.

Let Ω be a general, possibly non smooth, open set in $\mathbb{R}^n$. For any real $s > 0$ and for any $p \in [1, \infty)$, we want to define the fractional Sobolev spaces $W^{s,p}(\Omega)$. In the literature, fractional Sobolev-type spaces are also called *Aronszajn*, *Gagliardo* or *Slobodeckij spaces*, by the name of the ones who introduced them, almost simultaneously (see [3, 40, 79]).

We start by fixing the fractional exponent s in $(0, 1)$. For any $p \in [1, +\infty)$, we define $W^{s,p}(\Omega)$ as follows

$$(2.1) \qquad W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \ : \ \frac{|u(x) - u(y)|}{|x - y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\};$$

i.e, an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$(2.2) \qquad \|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p \, dx \ + \ \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx \, dy \right)^{\frac{1}{p}},$$

where the term

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx \, dy \right)^{\frac{1}{p}},$$

is the so-called *Gagliardo (semi)norm* of u.

It is worth noticing that, as in the classical case with s being an integer, the space $W^{s',p}$ is continuously embedded in $W^{s,p}$ when $s \le s'$, as next result points out.

Proposition 2.1. *Let $p \in [1, +\infty)$ and $0 < s \le s' < 1$. Let Ω be an open set in $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a measurable function. Then*

$$\|u\|_{W^{s,p}(\Omega)} \le C \|u\|_{W^{s',p}(\Omega)}$$

for some suitable positive constant $C = C(n, s, p) \ge 1$. In particular,

$$W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega).$$

Proof. First,

$$\int_\Omega \int_{\Omega \cap \{|x-y| \ge 1\}} \frac{|u(x)|^p}{|x - y|^{n+sp}} \, dx \, dy \ \le \ \int_\Omega \left(\int_{|z| \ge 1} \frac{1}{|z|^{n+sp}} \, dz \right) |u(x)|^p \, dx$$

$$\le \ C(n, s, p) \|u\|_{L^p(\Omega)}^p,$$

where we used the fact that the kernel $1/|z|^{n+sp}$ is integrable since $n + sp > n$.

Taking into account the above estimate, it follows

$$\int_\Omega \int_{\Omega \cap \{|x-y| \ge 1\}} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \ \le \ 2^{p-1} \int_\Omega \int_{\Omega \cap \{|x-y| \ge 1\}} \frac{|u(x)|^p + |u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy$$

$$(2.3) \qquad\qquad\qquad\qquad\qquad\qquad \le \ 2^p C(n, s, p) \|u\|_{L^p(\Omega)}^p.$$

On the other hand,

(2.4)
$$\int_\Omega \int_{\Omega \cap \{|x-y|<1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \leq \int_\Omega \int_{\Omega \cap \{|x-y|<1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+s'p}} \, dx \, dy \,.$$

Thus, combining (2.3) with (2.4), we get

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \leq 2^p C(n,s,p) \|u\|_{L^p(\Omega)}^p + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+s'p}} \, dx \, dy$$

and so

$$\|u\|_{W^{s,p}(\Omega)}^p \leq \left(2^p C(n,s,p)+1\right)\|u\|_{L^p(\Omega)}^p + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+s'p}} \, dx \, dy$$

$$\leq C(n,s,p)\|u\|_{W^{s',p}(\Omega)}^p,$$

which gives the desired estimate, up to relabeling the constant $C(n,p,s)$. $\qquad\square$

We will show in the forthcoming Proposition 2.2 that the result in Proposition 2.1 holds also in the limit case, namely when $s'=1$, but for this we have to take into account the regularity of $\partial\Omega$ (see Example 9.1).

As usual, for any $k \in \mathbb{N}$ and $\alpha \in (0,1]$, we say that Ω is of class $C^{k,\alpha}$ if there exists $M>0$ such that for any $x \in \partial\Omega$ there exists a ball $B=B_r(x)$, $r>0$, and an isomorphism $T:Q \to B$ such that

$$T \in C^{k,\alpha}(\overline{Q}), \quad T^{-1} \in C^{k,\alpha}(\overline{B}), \quad T(Q_+) = B \cap \Omega, \quad T(Q_0) = B \cap \partial\Omega$$

$$\text{and} \quad \|T\|_{C^{k,\alpha}(\overline{Q})} + \|T^{-1}\|_{C^{k,\alpha}(\overline{B})} \leq M,$$

where

$$Q := \left\{ x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,:\, |x'| < 1 \text{ and } |x_n| < 1 \right\},$$

$$Q_+ := \left\{ x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,:\, |x'| < 1 \text{ and } 0 < x_n < 1 \right\}$$

$$\text{and} \quad Q_0 := \left\{ x \in Q \,:\, x_n = 0 \right\}.$$

We have the following result.

Proposition 2.2. *Let $p \in [1, +\infty)$ and $s \in (0,1)$. Let Ω be an open set in $\mathbb{R}^n$ of class $C^{0,1}$ with bounded boundary and $u : \Omega \to \mathbb{R}$ be a measurable function. Then*

(2.5)
$$\|u\|_{W^{s,p}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}$$

for some suitable positive constant $C = C(n,s,p) \geq 1$. In particular,

$$W^{1,p}(\Omega) \subseteq W^{s,p}(\Omega)\,.$$

Proof. Let $u \in W^{1,p}(\Omega)$. Thanks to the regularity assumptions on the domain Ω, we can extend u to a function $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ such that $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$ and $\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}$ for a suitable constant C (see, e.g., [45, Theorem 7.25]).

Now, using the change of variable $z = y - x$ and the Hölder inequality, we have

$$
\begin{aligned}
\int_\Omega \int_{\Omega \cap \{|x-y|<1\}} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy \;\; &\leq\;\; \int_\Omega \int_{B_1} \frac{|u(x) - u(z+x)|^p}{|z|^{n+sp}}\, dz\, dx \\
&=\;\; \int_\Omega \int_{B_1} \frac{|u(x) - u(z+x)|^p}{|z|^p} \frac{1}{|z|^{n+(s-1)p}}\, dz\, dx \\
&\leq\;\; \int_\Omega \int_{B_1} \left(\int_0^1 \frac{|\nabla u(x + tz)|}{|z|^{\frac{n}{p}+s-1}}\, dt \right)^p dz\, dx \\
&\leq\;\; \int_{\mathbb{R}^n} \int_{B_1} \int_0^1 \frac{|\nabla \tilde{u}(x + tz)|^p}{|z|^{n+p(s-1)}}\, dt\, dz\, dx \\
&\leq\;\; \int_{B_1} \int_0^1 \frac{\|\nabla \tilde{u}\|_{L^p(\mathbb{R}^n)}^p}{|z|^{n+p(s-1)}}\, dt\, dz \\
&\leq\;\; C_1(n,s,p)\|\nabla \tilde{u}\|_{L^p(\mathbb{R}^n)}^p \\
(2.6) \qquad\qquad\qquad\qquad &\leq\;\; C_2(n,s,p)\|u\|_{W^{1,p}(\Omega)}^p\, .
\end{aligned}
$$

Also, by (2.3),

$$
(2.7) \qquad \int_\Omega \int_{\Omega \cap \{|x-y|\geq 1\}} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy \;\; \leq \;\; C(n,s,p)\|u\|_{L^p(\Omega)}^p\, .
$$

Therefore, from (2.6) and (2.7) we get estimate (2.5). $\qquad\qquad\qquad \square$

We remark that the Lipschitz assumption in Proposition 2.2 cannot be completely dropped (see Example 9.1 in Section 9); we also refer to the forthcoming Section 5, in which we discuss the extension problem in $W^{s,p}$.

Let us come back to the definition of the space $W^{s,p}(\Omega)$. Before going ahead, it is worth explaining why the definition in (2.1) cannot be plainly extended to the case $s \geq 1$. Suppose that Ω is a connected open set in $\mathbb{R}^n$, then any measurable function $u : \Omega \to \mathbb{R}$ such that

$$
\int_\Omega \int_\Omega \frac{|u(x) - u(u)|^p}{|x-y|^{n+sp}}\, dx\, dy \;<\; +\infty
$$

is actually constant (see [9, Proposition 2]). This fact is strictly related to the following result

$$(2.8) \qquad \lim_{s \to 1^-} (1-s)^{1/p} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \; = \; C \int_\Omega |\nabla u|^p \, dx$$

for a suitable positive constant C (see [9, Corollary 4]).

When $s > 1$ and it is not an integer we write $s = m + \sigma$, where m is an integer and $\sigma \in (0, 1)$. In this case the space $W^{s,p}(\Omega)$ consists of those equivalence classes of functions $u \in W^{m,p}(\Omega)$ whose distributional derivatives $D^\alpha u$, with $|\alpha| = m$, belong to $W^{\sigma,p}(\Omega)$, namely

$$(2.9) \qquad W^{s,p}(\Omega) := \left\{ u \in W^{m,p}(\Omega) \; : \; D^\alpha u \in W^{\sigma,p}(\Omega) \text{ for any } \alpha \text{ s.t. } |\alpha| = m \right\}$$

and this is a Banach space with respect to the norm

$$(2.10) \qquad \|u\|_{W^{s,p}(\Omega)} := \left(\|u\|_{W^{m,p}(\Omega)}^p + \sum_{|\alpha|=m} \|D^\alpha u\|_{W^{\sigma,p}(\Omega)}^p \right)^{\frac{1}{p}}.$$

Clearly, if $s = m$ is an integer, the space $W^{s,p}(\Omega)$ coincides with the Sobolev space $W^{m,p}(\Omega)$.

Corollary 2.3. *Let $p \in [1, +\infty)$ and $s, s' > 1$. Let Ω be an open set in $\mathbb{R}^n$ of class $C^{0,1}$. Then, if $s' \geq s$, we have*

$$W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega).$$

Proof. We write $s = k + \sigma$ and $s' = k' + \sigma'$, with k, k' integers and $\sigma, \sigma' \in (0, 1)$. In the case $k' = k$, we can use Proposition 2.1 in order to conclude that $W^{s',p}(\Omega)$ is continuously embedded in $W^{s,p}(\Omega)$. On the other hand, if $k' \geq k + 1$, using Proposition 2.1 and Proposition 2.2 we have the following chain

$$W^{k'+\sigma',p}(\Omega) \subseteq W^{k',p}(\Omega) \subseteq W^{k+1,p}(\Omega) \subseteq W^{k+\sigma,p}(\Omega).$$

The proof is complete. $\qquad\qquad\square$

As in the classic case with s being an integer, any function in the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ can be approximated by a sequence of smooth functions with compact support.

Theorem 2.4. *For any $s > 0$, the space $C_0^\infty(\mathbb{R}^n)$ of smooth functions with compact support is dense in $W^{s,p}(\mathbb{R}^n)$.*

A proof can be found in [1, Theorem 7.38].

Let $W_0^{s,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{W^{s,p}(\Omega)}$ defined in (2.10). Note that, in view of Theorem 2.4, we have

$$(2.11) \qquad W_0^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n),$$

but in general, for $\Omega \subset \mathbb{R}^n$, $W^{s,p}(\Omega) \neq W_0^{s,p}(\Omega)$, i.e. $C_0^\infty(\Omega)$ is not dense in $W^{s,p}(\Omega)$. Furthermore, it is clear that the same inclusions stated in Proposition 2.1, Proposition 2.2 and Corollary 2.3 hold for the spaces $W_0^{s,p}(\Omega)$.

Remark 2.5. For $s < 0$ and $p \in (1, \infty)$, we can define $W^{s,p}(\Omega)$ as the dual space of $W_0^{-s,q}(\Omega)$ where $1/p + 1/q = 1$. Notice that, in this case, the space $W^{s,p}(\Omega)$ is actually a space of distributions on Ω, since it is the dual of a space having $C_0^\infty(\Omega)$ as density subset.

Finally, it is worth noticing that the fractional Sobolev spaces play an important role in the trace theory. Precisely, for any $p \in (1, +\infty)$, assume that the open set $\Omega \subseteq \mathbb{R}^n$ is sufficiently smooth, then the space of traces Tu on $\partial\Omega$ of u in $W^{1,p}(\Omega)$ is characterized by $\|Tu\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} < +\infty$ (see [39]). Moreover, the trace operator T is surjective from $W^{1,p}(\Omega)$ onto $W^{1-\frac{1}{p},p}(\partial\Omega)$. In the quadratic case $p = 2$, the situation simplifies considerably, as we will see in the next two sections and a proof of the above trace embedding can be find in the forthcoming Proposition 4.5.

3. THE SPACE H^s AND THE FRACTIONAL LAPLACIAN OPERATOR

In this section, we focus on the case $p = 2$. This is quite an important case since the fractional Sobolev spaces $W^{s,2}(\Omega)$ and $W_0^{s,2}(\Omega)$ turn out to be Hilbert spaces. They are usually denoted by $H^s(\Omega)$ and $H_0^s(\Omega)$, respectively. Moreover, they are strictly related to the fractional Laplacian operator $(-\Delta)^s$ (see Proposition 4.4), where, for any $u \in \mathscr{S}$ and $s \in (0, 1)$, $(-\Delta)^s$ is defined as
(3.1)
$$(-\Delta)^s u(x) = C(n,s) P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy = C(n,s) \lim_{\varepsilon \to 0^+} \int_{\mathscr{C}B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy.$$

Here $P.V.$ is a commonly used abbreviation for "in the principal value sense" (as defined by the latter equation) and $C(n,s)$ is a dimensional constant that depends on s, precisely given by

(3.2) $$C(n,s) = \pi^{-(2s+n/2)} \frac{\Gamma(n/2 + s)}{\Gamma(-s)}$$

(see Remark 3.4).

Note that the factor $C(n,s)$ degenerates when $s \to 1$ since the *Gamma function* is a meromorphic function with simple pole at each negative integer.

Remark 3.1. Due to the singularity of the kernel, the right hand-side of (3.1) is not well defined in general. In the case $s \in (0, 1/2)$ the integral in (3.1) is not really

singular near x. Indeed, for any $u \in \mathscr{S}$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|}{|x - y|^{n+2s}} \, dy & \leq C \int_{B_R} \frac{|x - y|}{|x - y|^{n+2s}} \, dy + \|u\|_{L^\infty(\mathbb{R}^n)} \int_{\mathscr{C}B_R} \frac{1}{|x - y|^{n+2s}} \, dy \\
& = C \left(\int_{B_R} \frac{1}{|x - y|^{n+2s-1}} \, dy + \int_{\mathscr{C}B_R} \frac{1}{|x - y|^{n+2s}} \, dy \right) \\
& = C \left(\int_0^R \frac{1}{|\rho|^{2s}} \, d\rho + \int_R^{+\infty} \frac{1}{|\rho|^{2s+1}} \, d\rho \right) < +\infty
\end{aligned}
$$

where C is a positive constant depending only on the dimension and on the L^∞ norm of u.

In the following, we will show that we can equivalently define the fractional Laplacian operator as a Riesz potential of negative order.

The *Riesz kernel* I_α, with $\alpha \in (0, n)$, is defined by

(3.3) $$ I_\alpha(x) := \frac{1}{\gamma(n, \alpha)} |x|^{\alpha-n} $$

where

(3.4) $$ \gamma(n, \alpha) = \pi^{\frac{n}{2}-\alpha} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}. $$

The *Riesz potential* $\mathscr{I}_\alpha(f)$ is defined as the convolution with the Riesz Kernel, that is

(3.5) $$ \mathscr{I}_\alpha(f) := I_\alpha * f(x) = \frac{1}{\gamma(n, \alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy. $$

The value of the normalization factor $\gamma(n, \alpha)^{-1}$ in (3.3) is chosen in a suitable way in order to simplify computation via Fourier transform, as stated in the following proposition.

Proposition 3.2. *Let $\alpha \in (0, n)$. Then*

(3.6) $$ \mathscr{F}(I_\alpha(x)) = |y|^{-\alpha}, $$

where the equality is intended in the distributional sense.

Proof. We want to prove that

(3.7) $$ \int_{\mathbb{R}^n} |y|^{-\alpha} \varphi(y) \, dy = \frac{1}{\gamma(n, \alpha)} \int_{\mathbb{R}^n} |x|^{\alpha-n} \mathscr{F}\varphi(x) \, dx $$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$.

Since $\mathscr{F}(e^{-\pi\delta|x|^2}) = \delta^{-n/2} e^{-\pi|y|^2/\delta}$, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have that

$$
\int_{\mathbb{R}^n} \delta^{-\frac{n}{2}} e^{-\frac{\pi|y|^2}{\delta}} \varphi(y) \, dy = \int_{\mathbb{R}^n} e^{-\pi\delta|x|^2} \mathscr{F}\varphi(x) \, dx.
$$

Therefore, multiplying by $\delta^{\frac{n-\alpha-2}{2}}$ and integrating with respect to $\delta \in (0, +\infty)$, we obtain

$$(3.8) \quad \int_0^{+\infty} \int_{\mathbb{R}^n} \delta^{-\frac{(\alpha+2)}{2}} e^{-\frac{\pi|y|^2}{\delta}} \varphi(y) \, dy \, d\delta \; = \; \int_0^{+\infty} \int_{\mathbb{R}^n} \delta^{\frac{n-\alpha-2}{2}} e^{-\pi\delta|x|^2} \mathscr{F}\varphi(x) \, dx \, d\delta.$$

Let us consider separately the left and right hand-side of the above equality. On the left hand-side, by using the changing variable formula (with $t = |y|^2/\delta$), we get

$$\int_0^{+\infty} \int_{\mathbb{R}^n} \delta^{-\frac{(\alpha+2)}{2}} e^{-\frac{\pi|y|^2}{\delta}} \varphi(y) \, dy \, d\delta \; = \; \int_0^{+\infty} \int_{\mathbb{R}^n} |y|^{-\alpha} t^{\frac{(\alpha-2)}{2}} e^{-\pi t} \varphi(y) \, dy \, dt$$

$$(3.9) \hspace{6cm} = \; A \int_{\mathbb{R}^n} |y|^{-\alpha} \varphi(y) \, dy$$

where $A := \displaystyle\int_0^{+\infty} t^{\frac{(\alpha-2)}{2}} e^{-\pi t} \, dt$.

Similarly, by setting $\tau = |x|^2 \delta$ on the right-hand side of (3.8), we get

$$\int_0^{+\infty} \int_{\mathbb{R}^n} \delta^{\frac{n-\alpha-2}{2}} e^{-\pi\delta|x|^2} \mathscr{F}\varphi(x) \, dx \, d\delta \; = \; \int_0^{+\infty} \int_{\mathbb{R}^n} |x|^{\alpha-n} \tau^{\frac{(n-\alpha-2)}{2}} e^{-\pi\tau} \mathscr{F}\varphi(x) \, dx \, d\tau$$

$$(3.10) \hspace{6cm} = \; B \int_{\mathbb{R}^n} |x|^{\alpha-n} \mathscr{F}\varphi(x) \, dx$$

where $B := \displaystyle\int_0^{+\infty} \tau^{\frac{(n-\alpha-2)}{2}} e^{-\pi\tau} \, d\tau$.

Finally, we obtain (3.7) by combining (3.9) with (3.10), if we show that

$$A/B \; = \; \gamma(n, \alpha).$$

The above identity follows by the definition of the Gamma function together with a simple changing of variable. Indeed,

$$(3.11) \hspace{2cm} A \; = \; \int_0^{+\infty} t^{\frac{(\alpha-2)}{2}} e^{-\pi t} \, dt \; = \; \pi^{-\frac{\alpha}{2}} \int_0^{+\infty} \eta^{\frac{\alpha}{2}-1} e^{-\eta} \, d\eta \; = \; \pi^{-\frac{\alpha}{2}} \Gamma\left(\frac{\alpha}{2}\right)$$

and
$$(3.12)$$
$$B \; = \; \int_0^{+\infty} \tau^{\frac{(n-\alpha-2)}{2}} e^{-\pi\tau} \, d\tau \; = \; \pi^{\frac{-n+\alpha}{2}} \int_0^{+\infty} \eta^{\frac{(n-\alpha)}{2}-1} e^{-\eta} \, d\eta \; = \; \pi^{\frac{n-\alpha}{2}} \Gamma\left(\frac{n-\alpha}{2}\right).$$

Therefore, combining (3.11) with (3.12), we get the desired identity. $\qquad\square$

Remark 3.3. Note that for $\alpha = 2$ and $n > 2$, the Riesz Kernel I_2 defined in (3.3) coincides with the *Newtonian kernel* and it also satisfies the equation

$$-\Delta I_2(x) \; = \; -\Delta * I_2(x) \; = \; 4\pi^2 \delta.$$

Indeed, since $\Gamma(z+1) = z\,\Gamma(z)$ for any complex number $z \notin \{0, -1, -2, \cdots\}$ and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$, where ω_n denotes the measure of the unit sphere in $\mathbb{R}^n$,

$$I_2(x) = \pi^{2-\frac{n}{2}} \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(1)} |x|^{2-n} = \frac{2\pi^2}{\pi^{n/2}} \frac{\Gamma(n/2)}{(n-2)} |x|^{2-n} = \frac{4\pi^2}{(n-2)} \frac{1}{\omega_n} |x|^{2-n}.$$

Now, for any function $f \in \mathscr{S}$, we can consider the operator

$$\mathscr{I}_{2s}(f) = (-\Delta)^{-s} f, \qquad s \in (0, n/2).$$

The fractional Laplacian, for any $s \in (0, 1)$ may be also defined (see [44]) as

$$(-\Delta)^s u = -\Delta\left(\mathscr{I}_{2-2s}(u)\right)$$

and, therefore, by the properties of the Riesz operator (see, e.g., [53]), it follows

$$(3.13) \qquad\qquad (-\Delta)^s u = -\mathscr{I}_{-2s}(u).$$

For this reason the fractional Laplacian is also called the Riesz fractional derivative.

Remark 3.4. We are in position to justify the choice of the constant factor $C(n, s)$ in (3.1) and in (3.2). First, we note that $C(n, s)$ does not depend on the function u. Hence, we can take a function u such that $u(x) = 0$ for some $x \in \mathbb{R}^n$ and we get

$$(-\Delta)^s u(x) = -C(n, s)\, P.V. \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n+2s}}\,dy = -C(n, s)\,\gamma(n, -2s)\mathscr{I}_{-2s}(u).$$

By taking into account (3.13), the definition of $C(n, s)$ in (3.2) plainly follows.

We conclude this section by showing that one may write the singular integral in (3.1) as a weighted second order differential quotient.

Lemma 3.5. *Let $s \in (0, 1)$ and let $(-\Delta)^s$ be the fractional Laplacian operator defined by (3.1). Then, for any $u \in \mathscr{S}$,*

$$(3.14) \qquad (-\Delta)^s u(x) = -\frac{1}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|x-y|^{n+2s}}\,dy, \quad \forall x \in \mathbb{R}^n.$$

Proof. The equivalence of the definitions in (3.1) and (3.14) immediately follows by the standard changing variable formula.

Indeed, by choosing $z = y - x$, we have

$$(3.15) \quad (-\Delta)^s u(x) = -P.V. \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|x-y|^{n+2s}}\,dy = -P.V. \int_{\mathbb{R}^n} \frac{u(x+z) - u(x)}{|z|^{n+2s}}\,dz.$$

Moreover, by substituting $\tilde{z} = -z$ in last term of the above equality, we have

$$(3.16) \qquad P.V. \int_{\mathbb{R}^n} \frac{u(x+z) - u(x)}{|z|^{n+2s}}\,dz = P.V. \int_{\mathbb{R}^n} \frac{u(x-\tilde{z}) - u(x)}{|\tilde{z}|^{n+2s}}\,d\tilde{z}.$$

and so after relabeling $\tilde{z}$ as z

$$
\begin{aligned}
2P.V.\int_{\mathbb{R}^n} \frac{u(x+z)-u(x)}{|z|^{n+2s}}\,dz \;=\;& P.V.\int_{\mathbb{R}^n} \frac{u(x+z)-u(x)}{|z|^{n+2s}}\,dz \\
&+P.V.\int_{\mathbb{R}^n} \frac{u(x-z)-u(x)}{|z|^{n+2s}}\,dz \\
=\;& P.V.\int_{\mathbb{R}^n} \frac{u(x+z)+u(x-z)-2u(x)}{|z|^{n+2s}}\,dz.
\end{aligned}
$$
(3.17)

Therefore, if we rename z as y in (3.15) and (3.17), we can write the fractional Laplacian operator in (3.1) as

$$
(-\Delta)^s u(x) \;=\; -\frac{1}{2}\,P.V.\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy.
$$

The above representation is useful to remove the singularity of the integral at the origin. Indeed, for any smooth function u, a second order Taylor expansion yields

$$
\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}} \;\leq\; \frac{\|D^2 u\|_{L^\infty}}{|y|^{n+2s-2}},
$$

which is integrable near 0 (for any fixed $s \in (0,1)$). Therefore, since $u \in \mathscr{S}$, one can get rid of the $P.V.$ and write (3.14). $\qquad \square$

4. An approach via the Fourier transform

In this section, we fix $p = 2$ and we take into account an alternative definition of the space $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ via the Fourier transform. Precisely, we may define

$$
\hat{H}^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \; : \; \int_{\mathbb{R}^n} (1+|\xi|^{2s})|\mathscr{F}u(\xi)|^2\,d\xi < +\infty \right\}
$$
(4.1)

and we observe that the above definition, unlike the ones via the Gagliardo norm in (2.2), is valid also for any real $s \geq 1$.

We may also use an analogous definition for the case $s < 0$ by setting

$$
\hat{H}^s(\mathbb{R}^n) = \left\{ u \in \mathscr{S}'(\mathbb{R}^n) \; : \; \int_{\mathbb{R}^n} (1+|\xi|^2)^s|\mathscr{F}u(\xi)|^2\,d\xi < +\infty \right\},
$$

although in this case the space $\hat{H}^s(\mathbb{R}^n)$ is not a subset of $L^2(\mathbb{R}^n)$ and, in order to use the Fourier transform, one has to start from an element of $\mathscr{S}'(\mathbb{R}^n)$, (see also Remark 2.5).

The equivalence of the space $\hat{H}^s(\mathbb{R}^n)$ defined in (4.1) with the one defined in the previous section via the Gagliardo norm (see (2.1)) is stated and proven in the forthcoming Proposition 4.2.

First, we will prove that the fractional Laplacian $(-\Delta)^s$ can be viewed as a pseudo-differential operator of symbol $|\xi|^{2s}$. The proof is standard and it can be found in many papers (see, for instance, [81, Chapter 16]). We will follow the one

in [87] (see Section 3), in which is shown how singular integrals naturally arise as a continuous limit of discrete long jump random walks.

Proposition 4.1. *Let $s \in (0,1)$ and let $(-\Delta)^s : \mathscr{S} \to L^2(\mathbb{R}^n)$ be the fractional Laplacian operator defined by* (3.1). *Then, for any $u \in \mathscr{S}$,*

$$(4.2) \qquad\qquad (-\Delta)^s u = C \mathscr{F}^{-1}(|\xi|^{2s}(\mathscr{F}u)) \quad \forall \xi \in \mathbb{R}^n,$$

for a suitable positive constant C depending only on s and n.

Proof. In view of Lemma 3.5, we may use the definition via the weighted second order differential quotient in (3.14). We denote by $\mathscr{L}u$ the integral in (3.14), that is

$$\mathscr{L}u(x) = -\frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy.$$

$\mathscr{L}$ is a linear operator and we are looking for its "symbol" (or "multiplier"), that is a function $\mathcal{S} : \mathbb{R}^n \to \mathbb{R}$ such that

$$(4.3) \qquad\qquad \mathscr{L}u = C\mathscr{F}^{-1}(\mathcal{S}(\mathscr{F}u)),$$

for a suitable positive constant C depending only on s and n.

We want to prove that

$$(4.4) \qquad\qquad \mathcal{S}(\xi) = |\xi|^{2s},$$

where we denoted by ξ the frequency variable. To this scope, we point out that

$$\frac{|u(x+y) + u(x-y) - 2u(x)|}{|y|^{n+2s}}$$

$$\leq 4\left(\chi_{B_1}(y)|y|^{2-n-2s}\sup_{B_1(x)}|D^2u| + \chi_{\mathbb{R}^n\setminus B_1}(y)|y|^{-n-2s}\sup_{\mathbb{R}^n}|u|\right)$$

$$\leq C\left(\chi_{B_1}(y)|y|^{2-n-2s}(1+|x|^{n+1})^{-1} + \chi_{\mathbb{R}^n\setminus B_1}(y)|y|^{-n-2s}\right) \in L^1(\mathbb{R}^{2n}).$$

Consequently, by the Fubini-Tonelli's Theorem, we can exchange the integral in y with the Fourier transform in x. Thus, we apply the Fourier transform in the variable x in (4.3) and we obtain

$$\mathcal{S}(\xi)(\mathscr{F}u)(\xi) = \mathscr{F}(\mathscr{L}u)$$

$$= -\frac{1}{2}\int_{\mathbb{R}^n} \frac{\mathscr{F}\left(u(x+y) + u(x-y) - 2u(x)\right)}{|y|^{n+2s}}\, dy$$

$$= -\frac{1}{2}\int_{\mathbb{R}^n} \frac{e^{i\xi\cdot y} + e^{-i\xi\cdot y} - 2}{|y|^{n+2s}}\, dy(\mathscr{F}u)(\xi)$$

$$(4.5) \qquad\qquad = \int_{\mathbb{R}^n} \frac{1 - \cos(\xi\cdot y)}{|y|^{n+2s}}\, dy(\mathscr{F}u)(\xi).$$

Hence, in order to obtain (4.4), it suffices to show that

$$(4.6) \qquad \int_{\mathbb{R}^n} \frac{1 - \cos(\xi \cdot y)}{|y|^{n+2s}} \, dy = C|\xi|^{2s}.$$

To check this, first we observe that, if $\zeta = (\zeta_1, ..., \zeta_n) \in \mathbb{R}^n$, we have

$$\frac{1 - \cos \zeta_1}{|\zeta|^{n+2s}} \le \frac{|\zeta_1|^2}{|\zeta|^{n+2s}} \le \frac{1}{|\zeta|^{n-2+2s}}$$

near $\zeta = 0$. Thus,

$$(4.7) \qquad \int_{\mathbb{R}^n} \frac{1 - \cos \zeta_1}{|\zeta|^{n+2s}} \, d\zeta \quad \text{is finite and positive.}$$

Now, we consider the function $\mathcal{I} : \mathbb{R}^n \to \mathbb{R}$ defined as follows

$$\mathcal{I}(\xi) = \int_{\mathbb{R}^n} \frac{1 - \cos(\xi \cdot y)}{|y|^{n+2s}} dy.$$

We have that $\mathcal{I}$ is rotationally invariant, that is

$$(4.8) \qquad \mathcal{I}(\xi) = \mathcal{I}(|\xi|e_1),$$

where e_1 denotes the first direction vector in $\mathbb{R}^n$. Indeed, when $n = 1$, then we can deduce (4.8) by the fact that $\mathcal{I}(-\xi) = \mathcal{I}(\xi)$. When $n \ge 2$, we consider a rotation R for which $R(|\xi|e_1) = \xi$ and we denote by R^T its transpose. Then, by substituting $\tilde{y} = R^T y$, we obtain

$$\begin{aligned}
\mathcal{I}(\xi) &= \int_{\mathbb{R}^n} \frac{1 - \cos\left((R(|\xi|e_1)) \cdot y\right)}{|y|^{n+2s}} \, dy \\
&= \int_{\mathbb{R}^n} \frac{1 - \cos\left((|\xi|e_1) \cdot (R^T y)\right)}{|y|^{n+2s}} \, dy \\
&= \int_{\mathbb{R}^n} \frac{1 - \cos\left((|\xi|e_1) \cdot \tilde{y}\right)}{|\tilde{y}|^{n+2s}} \, d\tilde{y} = \mathcal{I}(|\xi|e_1),
\end{aligned}$$

which proves (4.8).

As a consequence of (4.7) and (4.8), the substitution $\zeta = |\xi|y$ gives that

$$\begin{aligned}
\mathcal{I}(\xi) &= \mathcal{I}(|\xi|e_1) \\
&= \int_{\mathbb{R}^n} \frac{1 - \cos(|\xi|y_1)}{|y|^{n+2s}} \, dy \\
&= \frac{1}{|\xi|^n} \int_{\mathbb{R}^n} \frac{1 - \cos \zeta_1}{|\zeta/|\xi||^{n+2s}} \, d\zeta = C|\xi|^{2s}.
\end{aligned}$$

Hence, we deduce (4.6) and then the proof is complete. $\qquad\qquad\square$

Proposition 4.2. *Let $s \in (0,1)$. Then the fractional Sobolev space $H^s(\mathbb{R}^n)$ defined in Section 2 coincides with the space $\hat{H}^s(\mathbb{R}^n)$ defined in (4.1). In particular, for any $u \in H^s(\mathbb{R}^n)$*

$$[u]_{H^s(\mathbb{R}^n)} = C \int_{\mathbb{R}^n} |\xi|^{2s} |\mathscr{F}u(\xi)|^2 \, d\xi,$$

for a suitable positive constant C depending only on n and s.

Proof. For every fixed $y \in \mathbb{R}^n$, by changing of variable choosing $z = x - y$, we get

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \right) dy = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(z+y) - u(y)|^2}{|z|^{n+2s}} \, dz \, dy$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left| \frac{u(z+y) - u(y)}{|z|^{n/2+s}} \right|^2 dy \right) dz$$

$$= \int_{\mathbb{R}^n} \left\| \frac{u(z+\cdot) - u(\cdot)}{|z|^{n/2+s}} \right\|_{L^2(\mathbb{R}^n)}^2 dz$$

$$= \int_{\mathbb{R}^n} \left\| \mathscr{F}\left(\frac{u(z+\cdot) - u(\cdot)}{|z|^{n/2+s}} \right) \right\|_{L^2(\mathbb{R}^n)}^2 dz,$$

where Plancherel Formula has been used.

Now, using (4.6) we obtain

$$\int_{\mathbb{R}^n} \left\| \mathscr{F}\left(\frac{u(z+\cdot) - u(\cdot)}{|z|^{n/2+s}} \right) \right\|_{L^2(\mathbb{R}^n)}^2 dz = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|e^{i\xi \cdot z} - 1|^2}{|z|^{n+2s}} |\mathscr{F}u(\xi)|^2 \, d\xi \, dz$$

$$= 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(1 - \cos \xi \cdot z)}{|z|^{n+2s}} |\mathscr{F}u(\xi)|^2 \, dz \, d\xi$$

$$= 2C_1 \int_{\mathbb{R}^n} |\xi|^{2s} |\mathscr{F}u(\xi)|^2 \, d\xi,$$

where the constant C_1 is equal to $\displaystyle\int_{\mathbb{R}^n} \frac{1 - \cos(\zeta_1)}{|\zeta|^{n+2s}} \, d\zeta$. This completes the proof. $\square$

Remark 4.3. The equivalence of the spaces H^s and $\hat{H}^s$ stated in Proposition 4.2 relies on Plancherel Formula. As well known, unless $p = q = 2$, one cannot go forward and backward between an L^p and an L^q via Fourier transform (see, for instance, the sharp inequality in [5] for the case $1 < p < 2$ and q equal to the conjugate exponent $p/(p-1)$). That is why the general fractional space defined via Fourier transform for $1 < p < \infty$ and $s > 0$, say $H^{s,p}(\mathbb{R}^n)$, does not coincide with the fractional Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ and will be not discussed here (see, e.g., [90]).

Finally, we are able to prove the relation between the fractional Laplacian operator $(-\Delta)^s$ and the fractional Sobolev space H^s.

Proposition 4.4. *Let $s \in (0,1)$ and let $u \in H^s(\mathbb{R}^n)$. Then,*

$$(4.9) \qquad [u]_{H^s(\mathbb{R}^n)} = C\|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^n)},$$

for a suitable positive constant C depending only on s and n.

Proof. The equality in (4.9) plainly follows from Proposition 4.1 and Proposition 4.2. Indeed,

$$\|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^n)} = \|\mathscr{F}(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^n)} = C\||\xi|^s\mathscr{F}u\|_{L^2(\mathbb{R}^n)}$$

$$= C[u]_{H^s(\mathbb{R}^n)}.$$

$\square$

Armed with the definition of $H^s(\mathbb{R}^n)$ via the Fourier transform, we can easily analyze the traces of the Sobolev functions (see the forthcoming Proposition 4.5). We will follow Sections 13, 15 and 16 in [81].

Let $\Omega \subseteq \mathbb{R}^n$ be an open set with continuous boundary $\partial\Omega$. Denote by T the *trace operator*, namely the linear operator defined by the uniformly continuous extension of the operator of restriction to $\partial\Omega$ for functions in $\mathcal{D}(\overline{\Omega})$, that is the space of functions $C_0^\infty(\mathbb{R}^n)$ restricted[2] to $\overline{\Omega}$.

Now, for any $x = (x', x_n) \in \mathbb{R}^n$ and for any $u \in \mathscr{S}(\mathbb{R}^n)$, we denote by $v \in \mathscr{S}(\mathbb{R}^{n-1})$ the restriction of u on the hyperplane $x_n = 0$, that is

$$(4.10) \qquad v(x') = u(x',0) \quad \forall x' \in \mathbb{R}^{n-1}.$$

Then, we have

$$(4.11) \qquad \mathscr{F}v(\xi') = \int_{\mathbb{R}} \mathscr{F}u(\xi', \xi_n)\, d\xi_n \quad \forall \xi' \in \mathbb{R}^{n-1},$$

where, for the sake of simplicity, we keep the same symbol $\mathscr{F}$ for both the Fourier transform in $n-1$ and in n variables.

To check (4.11), we write

$$\mathscr{F}v(\xi') = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-i\xi'\cdot x'} v(x')\, dx'$$

$$(4.12) \qquad\qquad = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-i\xi'\cdot x'} u(x',0)\, dx'.$$

---

[2] Notice that we cannot simply take T as the restriction operator to the boundary, since the restriction to a set of measure 0 (like the set $\partial\Omega$) is not defined for functions which are not smooth enough.

On the other hand, we have

$$\int_{\mathbb{R}} \mathscr{F}u(\xi', \xi_n)\, d\xi_n$$

$$= \int_{\mathbb{R}} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\,(\xi',\xi_n)\cdot(x',x_n)} u(x', x_n)\, dx'\, dx_n\, d\xi_n$$

$$= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-i\xi'\cdot x'} \left[\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\xi_n\cdot x_n} u(x', x_n)\, dx_n d\xi_n \right] dx'$$

$$= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-i\xi'\cdot x'} \big[u(x', 0) \big]\, dx',$$

where the last equality follows by transforming and anti-transforming u in the last variable, and this coincides with (4.12).

Now, we are in position to characterize the traces of the function in $H^s(\mathbb{R}^n)$, as stated in the following proposition.

Proposition 4.5. ([81, Lemma 16.1]). *Let $s > 1/2$, then any function $u \in H^s(\mathbb{R}^n)$ has a trace v on the hyperplane $\{x_n = 0\}$, such that $v \in H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$. Also, the trace operator T is surjective from $H^s(\mathbb{R}^n)$ onto $H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$.*

Proof. In order to prove the first claim, it suffices to show that there exists an universal constant C such that, for any $u \in \mathscr{S}(\mathbb{R}^n)$ and any v defined as in (4.10),

$$(4.13) \qquad \|v\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \leq C\|u\|_{H^s(\mathbb{R}^n)}.$$

By taking into account (4.11), the Cauchy-Schwarz inequality yields

$$(4.14) \qquad |\mathscr{F}v(\xi')|^2 \leq \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s |\mathscr{F}u(\xi', \xi_n)|^2\, d\xi_n \right) \left(\int_{\mathbb{R}} \frac{d\xi_n}{(1 + |\xi|^2)^s} \right).$$

Using the changing of variable formula by setting $\xi_n = t\sqrt{1 + |\xi'|^2}$, we have

$$\int_{\mathbb{R}} \frac{d\xi_n}{(1 + |\xi|^2)^s} = \int_{\mathbb{R}} \frac{(1 + |\xi'|^2)^{1/2}}{((1 + |\xi'|^2)(1 + t^2))^s}\, dt = \int_{\mathbb{R}} \frac{(1 + |\xi'|)^{\frac{1}{2} - s}}{(1 + t^2)^s}\, dt$$

$$(4.15) \qquad = C(s)\big(1 + |\xi'|^2\big)^{\frac{1}{2} - s},$$

where $C(s) := \int_{\mathbb{R}} \frac{dt}{(1 + t^2)^s} < +\infty$ since $s > 1/2$.

Combining (4.14) with (4.15) and integrating in $\xi' \in \mathbb{R}^{n-1}$, we obtain

$$\int_{\mathbb{R}^{n-1}} \big(1 + |\xi'|^2\big)^{s - \frac{1}{2}} |\mathscr{F}v(\xi')|^2\, d\xi' \leq C(s) \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \big(1 + |\xi|^2\big)^s |\mathscr{F}u(\xi', \xi_n)|^2\, d\xi_n\, d\xi',$$

that is (4.13).

Now, we will prove the surjectivity of the trace operator T. For this, we show that for any $v \in H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ the function u defined by

$$(4.16) \qquad \mathscr{F}u(\xi', \xi_n) = \mathscr{F}v(\xi')\, \varphi\left(\frac{\xi_n}{\sqrt{1+|\xi'|^2}}\right) \frac{1}{\sqrt{1+|\xi'|^2}},$$

with $\varphi \in C_0^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} \varphi(t)\, dt = 1$, is such that $u \in H^s(\mathbb{R}^n)$ and $Tu = v$. Indeed, we integrate (4.16) with respect to $\xi_n \in \mathbb{R}$, we substitute $\xi_n = t\sqrt{1+|\xi'|^2}$ and we obtain

$$\int_{\mathbb{R}} \mathscr{F}u(\xi', \xi_n)\, d\xi_n \;=\; \int_{\mathbb{R}} \mathscr{F}v(\xi')\, \varphi\left(\frac{\xi_n}{\sqrt{1+|\xi'|^2}}\right) \frac{1}{\sqrt{1+|\xi'|^2}}\, d\xi_n$$

$$(4.17) \qquad\qquad\qquad = \int_{\mathbb{R}} \mathscr{F}v(\xi')\, \varphi(t)\, dt \;=\; \mathscr{F}v(\xi')$$

and this implies $v = Tu$ because of (4.11).

The proof of the H^s-boundedness of u is straightforward. In fact, from (4.16), for any $\xi' \in \mathbb{R}^{n-1}$, we have

$$\int_{\mathbb{R}} \left(1+|\xi|^2\right)^s |\mathscr{F}u(\xi', \xi_n)|^2\, d\xi_n$$

$$= \int_{\mathbb{R}} \left(1+|\xi|^2\right)^s |\mathscr{F}v(\xi')|^2 \left|\varphi\left(\frac{\xi_n}{\sqrt{1+|\xi'|^2}}\right)\right|^2 \frac{1}{1+|\xi'|^2}\, d\xi_n$$

$$(4.18) \qquad\qquad = C\left(1+|\xi'|^2\right)^{s-\frac{1}{2}} |\mathscr{F}v(\xi')|^2,$$

where we used again the changing of variable formula with $\xi_n = t\sqrt{1+|\xi'|^2}$ and the constant C is given by $\int_{\mathbb{R}} \left(1+t^2\right)^s |\varphi(t)|^2\, dt$. Finally, we obtain that $u \in H^s(\mathbb{R}^n)$ by integrating (4.18) in $\xi' \in \mathbb{R}^{n-1}$. $\square$

Remark 4.6. We conclude this section by recalling that the fractional Laplacian $(-\Delta)^s$, which is a nonlocal operator on functions defined in $\mathbb{R}^n$, may be reduced to a local, possibly singular or degenerate, operator on functions sitting in the higher dimensional half-space $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, +\infty)$. We have

$$(-\Delta)^s u(x) \;=\; -C \lim_{t \to 0} \left(t^{1-2s} \frac{\partial U}{\partial t}(x, t)\right),$$

where the function $U : \mathbb{R}_+^{n+1} \to \mathbb{R}$ solves $\mathrm{div}(t^{1-2s}\nabla U) = 0$ in $\mathbb{R}_+^{n+1}$ and $U(x,0) = u(x)$ in $\mathbb{R}^n$.

This approach was pointed out by Caffarelli and Silvestre in [17]; see, in particular, Section 3.2 there, where was also given an equivalent definition of the

$H^s(\mathbb{R}^n)$-norm:

$$\int_{\mathbb{R}^n} |\xi|^{2s} |\mathscr{F}u|^2 \, d\xi \,=\, C \int_{\mathbb{R}^{n+1}_+} |\nabla U|^2 t^{1-2s} \, dx \, dt.$$

The cited results turn out to be very fruitful in order to recover an elliptic PDE approach in a nonlocal framework, and they have recently been used very often (see, e.g., [16, 78, 11, 15], etc.).

5. EXTENDING A $W^{s,p}(\Omega)$ FUNCTION TO THE WHOLE OF $\mathbb{R}^n$

As well known when s is an integer, under certain regularity assumptions on the domain Ω, any function in $W^{s,p}(\Omega)$ may be extended to a function in $W^{s,p}(\mathbb{R}^n)$. Extension results are quite important in applications and are necessary in order to improve some embeddings theorems, in the classic case as well as in the fractional case (see Section 6 and Section 7 in the following).

For any $s \in (0,1)$ and any $p \in [1,\infty)$, we say that an open set $\Omega \subseteq \mathbb{R}^n$ is an *extension domain for $W^{s,p}$* if every function $u \in W^{s,p}(\Omega)$ can be extended to a function $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$ in a continuous way; i.e., there exists a constant $C = C(n, p, s, \Omega)$ such that $\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\Omega)}$.

In general, an arbitrary open set is not an extension domain for $W^{s,p}$. To the authors' knowledge, the problem of characterizing the class of sets that are extension domains for $W^{s,p}$ is open. The same happens when s is an integer, with the exception of the special case $s = 1$, $p = 2$ and $n = 2$ (see [52]). We refers the interested reader to the recent book by Leoni [54], in which this problem is very well discussed (see, in particular, Chapter 11 and Chapter 12 there).

In this section, we will show that any open set Ω of class $C^{0,1}$ with bounded boundary is an extension domain for $W^{s,p}$.

We start with some preliminary lemmas, in which we will construct the extension to the whole of $\mathbb{R}^n$ of a function u defined on Ω in two separated cases: when the function u is identically zero in a neighborhood of the boundary $\partial\Omega$ and when Ω coincides with the half-space $\mathbb{R}^n_+$.

Lemma 5.1. *Let Ω be an open set in $\mathbb{R}^n$ and u a function in $W^{s,p}(\Omega)$ with $s \in (0,1)$ and $p \in [1,+\infty)$. If there exists a compact subset $K \subset \Omega$ such that $u \equiv 0$ in $\Omega \setminus K$, then the extension function $\tilde{u}$ defined as*

(5.1)
$$\tilde{u}(x) = \begin{cases} u(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

belongs to $W^{s,p}(\mathbb{R}^n)$ and

$$\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\Omega)},$$

where C is a suitable positive constant depending on n, p, s, K and Ω.

Proof. Clearly $\tilde{u} \in L^p(\mathbb{R}^n)$. Hence, it remains to verify that the Gagliardo norm of $\tilde{u}$ in $\mathbb{R}^n$ is bounded by the one of u in Ω. Using the symmetry of the integral in the Gagliardo norm with respect to x and y and the fact that $\tilde{u} \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, we can split as follows

$$(5.2) \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \;=\; \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy$$
$$+ 2 \int_{\Omega} \left(\int_{\mathbb{R}^n \setminus \Omega} \frac{|u(x)|^p}{|x-y|^{n+sp}} \, dy \right) dx,$$

where the first term in the right hand-side of (5.2) is finite since $u \in W^{s,p}(\Omega)$. Furthermore, for any $y \in \mathbb{R}^n \setminus K$,

$$\frac{|u(x)|^p}{|x-y|^{n+sp}} = \frac{\chi_K(x)|u(x)|^p}{|x-y|^{n+sp}} \leq \chi_K(x)|u(x)|^p \sup_{x \in K} \frac{1}{|x-y|^{n+sp}}$$

and so

$$(5.3) \quad \int_{\Omega} \left(\int_{\mathbb{R}^n \setminus \Omega} \frac{|u(x)|^p}{|x-y|^{n+sp}} \, dy \right) dx \leq \int_{\mathbb{R}^n \setminus \Omega} \frac{1}{\mathrm{dist}(y, \partial K)^{n+sp}} \, dy \; \|u\|^p_{L^p(\Omega)} .$$

Note that the integral in (5.3) is finite since $\mathrm{dist}(\partial\Omega, \partial K) \geq \alpha > 0$ and $n + sp > n$. Combining (5.2) with (5.3), we get

$$\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{s,p}(\Omega)}$$

where $C = C(n, s, p, K)$. $\qquad \square$

Lemma 5.2. *Let Ω be an open set in $\mathbb{R}^n$, symmetric with respect to the coordinate x_n, and consider the sets $\Omega_+ = \{x \in \Omega : x_n > 0\}$ and $\Omega_- = \{x \in \Omega : x_n \leq 0\}$. Let u be a function in $W^{s,p}(\Omega_+)$, with $s \in (0,1)$ and $p \in [1, +\infty)$. Define*

$$(5.4) \qquad \bar{u}(x) = \begin{cases} u(x', x_n) & x_n \geq 0, \\ u(x', -x_n) & x_n < 0. \end{cases}$$

Then $\bar{u}$ belongs to $W^{s,p}(\Omega)$ and

$$\|\bar{u}\|_{W^{s,p}(\Omega)} \leq 4 \|u\|_{W^{s,p}(\Omega_+)} .$$

Proof. By splitting the integrals and changing variable $\hat{x} = (x', -x_n)$, we get

$$(5.5) \qquad \|\bar{u}\|^p_{L^p(\Omega)} \;=\; \int_{\Omega_+} |u(x)|^p \, dx + \int_{\Omega_+} |u(\hat{x}', \hat{x}_n)|^p \, d\tilde{x} \;=\; 2 \|u\|^p_{L^p(\Omega_+)}.$$

Also, if $x \in \mathbb{R}_+^n$ and $y \in \mathscr{C}\mathbb{R}_+^n$ then $(x_n - y_n)^2 \geq (x_n + y_n)^2$ and therefore

$$
\begin{aligned}
\int_\Omega \int_\Omega \frac{|\bar{u}(x) - \bar{u}(y)|^p}{|x - y|^{n+sp}} \, dx \, dy &= \int_{\Omega_+} \int_{\Omega_+} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \\
&+ 2 \int_{\Omega_+} \int_{\mathscr{C}\Omega_+} \frac{|u(x) - u(y', -y_n)|^p}{|x - y|^{n+sp}} \, dx \, dy \\
&+ \int_{\mathscr{C}\Omega_+} \int_{\mathscr{C}\Omega_+} \frac{|u(x', -x_n) - u(y', -y_n)|^p}{|x - y|^{n+sp}} \, dx \, dy \\
&\leq 4\|u\|_{W^{s,p}(\Omega_+)}^p.
\end{aligned}
$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Now, a truncation lemma near $\partial\Omega$.

Lemma 5.3. *Let Ω be an open set in $\mathbb{R}^n$, $s \in (0,1)$ and $p \in [1, +\infty)$. Let us consider $u \in W^{s,p}(\Omega)$ and $\psi \in C_0^\infty(\Omega)$, $0 \leq \psi \leq 1$. Then $\psi u \in W^{s,p}(\Omega)$ and*

$$
(5.6) \qquad\qquad \|\psi\, u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)},
$$

where $C = C(n, p, s, \Omega)$.

Proof. It is clear that $\|\psi\, u\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)}$ since $|\psi| \leq 1$. Furthermore, adding and subtracting the factor $\psi(x)u(y)$, we get

$$
\begin{aligned}
\int_\Omega \int_\Omega \frac{|\psi(x)\, u(x) - \psi(y)\, u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy &\leq \int_\Omega \int_\Omega \frac{|\psi(x)\, u(x) - \psi(x)\, u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \\
&+ \int_\Omega \int_\Omega \frac{|\psi(x)\, u(y) - \psi(y)\, u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \\
(5.7) \qquad &\leq \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \\
&+ \int_\Omega \int_\Omega \frac{|u(x)|^p \, |\psi(x) - \psi(y)|^p}{|x - y|^{n+sp}} \, dx \, dy.
\end{aligned}
$$

Since ψ is smooth, we have

$$
\begin{aligned}
\int_\Omega \int_\Omega \frac{|u(x)|^p \, |\psi(x) - \psi(y)|^p}{|x - y|^{n+sp}} \, dx \, dy &\leq \Lambda^p \int_\Omega \int_{\Omega \cap |x-y| \leq 1} \frac{|u(x)|^p \, |x - y|^p}{|x - y|^{n+sp}} \, dx \, dy \\
&+ \int_\Omega \int_{\Omega \cap |x-y| \geq 1} \frac{|u(x)|^p}{|x - y|^{n+sp}} \, dx \, dy \\
(5.8) \qquad &\leq \tilde{C}\|u\|_{L^p(\Omega)}^p,
\end{aligned}
$$

where Λ denotes the Lipschitz constant of ψ and $\tilde{C}$ is a positive constant depending on n, p and s. Note that the last inequality follows from the fact that the kernel $|x - y|^{-n+(1-s)p}$ is summable with respect to y if $|x - y| \leq 1$ since $n + (s - 1)p < n$

and, on the other hand, the kernel $|x - y|^{-n-sp}$ is summable when $|x - y| \geq 1$ since $n + sp > n$. Finally, combining (5.7) with (5.8), we obtain estimate (5.6). $\qquad \square$

Now, we are ready to prove the main theorem of this section, that states that every open Lipschitz set Ω with bounded boundary is an extension domain for $W^{s,p}$.

Theorem 5.4. *Let* $p \in [1, +\infty)$, $s \in (0, 1)$ *and* $\Omega \subseteq \mathbb{R}^n$ *be an open set of class* $C^{0,1}$ *with bounded boundary. Then* $W^{s,p}(\Omega)$ *is continuously embedded in* $W^{s,p}(\mathbb{R}^n)$, *namely for any* $u \in W^{s,p}(\Omega)$ *there exists* $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$ *such that* $\tilde{u}|_\Omega = u$ *and*

$$\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\Omega)}$$

where $C = C(n, p, s, \Omega)$.

Proof. Since $\partial\Omega$ is compact, we can find a finite number of balls B_j such that $\partial\Omega \subset \bigcup_{j=1}^{k} B_j$ and so we can write $\mathbb{R}^n = \bigcup_{j=1}^{k} B_j \cup (\mathbb{R}^n \setminus \partial\Omega)$.

If we consider this covering, there exists a partition of unity related to it, i.e. there exist k smooth functions $\psi_0, \psi_1, ..., \psi_k$ such that $\operatorname{spt}\psi_0 \subset \mathbb{R}^n \setminus \partial\Omega$, $\operatorname{spt}\psi_j \subset B_j$ for any $j \in \{1, ..., k\}$, $0 \leq \psi_j \leq 1$ for any $j \in \{0, ..., k\}$ and $\sum_{j=0}^{k} \psi_j = 1$. Clearly,

$$u = \sum_{j=0}^{k} \psi_j u.$$

By Lemma 5.3, we know that $\psi_0 u$ belongs to $W^{s,p}(\Omega)$. Furthermore, since $\psi_0 u \equiv 0$ in a neighborhood of $\partial\Omega$, we can extend it to the whole of $\mathbb{R}^n$, by setting

$$\widetilde{\psi_0 u}(x) = \begin{cases} \psi_0 u(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

and $\widetilde{\psi_0 u} \in W^{s,p}(\mathbb{R}^n)$. Precisely

(5.9) $\qquad \|\widetilde{\psi_0 u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\,\|\psi_0 u\|_{W^{s,p}(\Omega)} \leq C\,\|u\|_{W^{s,p}(\Omega)},$

where $C = C(n, s, p, \Omega)$ (possibly different step by step, see Lemma 5.1 and Lemma 5.3).

For any $j \in \{1, ..., k\}$, let us consider $u|_{B_j \cap \Omega}$ and set

$$v_j(y) := u\,(T_j(y)) \qquad \text{for any } y \in Q_+,$$

where $T_j : Q \to B$ is the isomorphism of class $C^{0,1}$ defined in Section 2. Note that such a T_j exists by the regularity assumption on the domain Ω.

Now, we state that $v_j \in W^{s,p}(Q_+)$. Indeed, using the standard changing variable formula by setting $x = T_j(\hat{x})$ we have

$$\int_{Q_+} \int_{Q_+} \frac{|v(\hat{x}) - v(\hat{y})|^p}{|\hat{x} - \hat{y}|^{n+sp}} \, d\hat{x} \, d\hat{y}$$

$$= \int_{Q_+} \int_{Q_+} \frac{|u(T_j(\hat{x})) - u(T_j(\hat{y}))|^p}{|\hat{x} - \hat{y}|^{n+sp}} \, d\hat{x} \, d\hat{y}$$

$$= \int_{B_j \cap \Omega} \int_{B_j \cap \Omega} \frac{|u(x) - u(y)|^p}{|T_j^{-1}(x) - T_j^{-1}(y)|^{n+sp}} \, \det(T_j^{-1}) dx \, dy$$

$$(5.10) \qquad = C \int_{B_j \cap \Omega} \int_{B_j \cap \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy,$$

where (5.10) follows from the fact that T_j is bi-Lipschitz. Moreover, using Lemma 5.2 we can extend v_j to all Q so that the extension $\bar{v}_j$ belongs to $W^{s,p}(Q)$ and

$$\|\bar{v}_j\|_{W^{s,p}(Q)} \le 4\|v_j\|_{W^{s,p}(Q_+)}.$$

We set

$$w_j(x) := \bar{v}_j\left(T_j^{-1}(x)\right) \quad \text{for any } x \in B_j.$$

Since T_j is bi-Lipschitz, by arguing as above it follows that $w_j \in W^{s,p}(B_j)$. Note that $w_j \equiv u$ (and consequently $\psi_j w_j \equiv \psi_j u$) on $B_j \cap \Omega$. By definition $\psi_j w_j$ has compact support in B_j and therefore, as done for $\psi_0 u$, we can consider the extension $\widetilde{\psi_j w_j}$ to all $\mathbb{R}^n$ in such a way that $\widetilde{\psi_j w_j} \in W^{s,p}(\mathbb{R}^n)$. Also, using Lemma 5.1, Lemma 5.2, Lemma 5.3 and estimate (5.10) we get

$$\|\widetilde{\psi_j w_j}\|_{W^{s,p}(\mathbb{R}^n)} \le C\|\psi_j w_j\|_{W^{s,p}(B_j)} \le C\|w_j\|_{W^{s,p}(B_j)}$$

$$\le C\|\bar{v}_j\|_{W^{s,p}(Q)} \le C\|v_j\|_{W^{s,p}(Q_+)}$$

$$(5.11) \qquad \le C\|u\|_{W^{s,p}(\Omega \cap B_j)},$$

where $C = C(n, p, s, \Omega)$ and it is possibly different step by step.

Finally, let

$$\tilde{u} = \widetilde{\psi_0 u} + \sum_{j=1}^{k} \widetilde{\psi_j w_j}$$

be the extension of u defined on all $\mathbb{R}^n$. By construction, it is clear that $\tilde{u}|_\Omega = u$ and, combining (5.9) with (5.11), we get

$$\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \le C\|u\|_{W^{s,p}(\Omega)}$$

with $C = C(n, p, s, \Omega)$. □

Corollary 5.5. *Let $p \in [1, +\infty)$, $s \in (0, 1)$ and Ω be an open set in $\mathbb{R}^n$ of class $C^{0,1}$ with bounded boundary. Then for any $u \in W^{s,p}(\Omega)$, there exists a sequence $\{u_n\} \in C_0^\infty(\mathbb{R}^n)$ such that $u_n \to u$ as $n \to +\infty$ in $W^{s,p}(\Omega)$, i.e.,*

$$\lim_{n \to +\infty} \|u_n - u\|_{W^{s,p}(\Omega)} = 0.$$

Proof. The proof follows directly by Theorem 2.4 and Theorem 5.4. □

6. Fractional Sobolev inequalities

In this section, we provide an elementary proof of a Sobolev-type inequality involving the fractional norm $\| \cdot \|_{W^{s,p}}$ (see Theorem 6.5 below).

The original proof is contained in the Appendix of [74] and it deals with the case $p = 2$ (see, in particular, Theorem 7 there). We note that when $p = 2$ and $s \in [1/2, 1)$ some of the statements may be strengthened (see [9]). We also note that more general embeddings for the spaces $W^{s,p}$ can be obtained by interpolation techniques and by passing through Besov spaces; see, for instance, [6, 7, 85, 86, 57]. For a more comprehensive treatment of fractional Sobolev-type inequalities we refer to [55, 56, 10, 1, 81] and the references therein.

We remark that the proof here is self-contained. Moreover, we will not make use of Besov or fancy interpolation spaces.

In order to prove the Sobolev-type inequality in forthcoming Theorem 6.5, we need some preliminary results. The first of them is an elementary estimate involving the measure of finite measurable sets E in $\mathbb{R}^n$ as stated in the following lemma (see [75, Lemma A.1]).

Lemma 6.1. *Fix $x \in \mathbb{R}^n$. Let $p \in [1, +\infty)$, $s \in (0, 1)$ and $E \subset \mathbb{R}^n$ be a measurable set with finite measure. Then,*

$$\int_{\mathscr{C}E} \frac{dy}{|x - y|^{n+sp}} \geq C |E|^{-sp/n},$$

for a suitable constant $C = C(n, p, s) > 0$.

Proof. We set

$$\rho := \left(\frac{|E|}{\omega_n} \right)^{\frac{1}{n}}$$

and then it follows

$$
\begin{aligned}
|(\mathscr{C}E) \cap B_\rho(x)| &= |B_\rho(x)| - |E \cap B_\rho(x)| = |E| - |E \cap B_\rho(x)| \\
&= |E \cap \mathscr{C}B_\rho(x)|.
\end{aligned}
$$

Therefore,

$$
\int_{\mathscr{C}E} \frac{dy}{|x-y|^{n+sp}} = \int_{(\mathscr{C}E)\cap B_\rho(x)} \frac{dy}{|x-y|^{n+sp}} + \int_{(\mathscr{C}E)\cap \mathscr{C}B_\rho(x)} \frac{dy}{|x-y|^{n+sp}}
$$

$$
\geq \int_{(\mathscr{C}E)\cap B_\rho(x)} \frac{dy}{\rho^{n+sp}} + \int_{(\mathscr{C}E)\cap \mathscr{C}B_\rho(x)} \frac{dy}{|x-y|^{n+sp}}
$$

$$
= \frac{|(\mathscr{C}E)\cap B_\rho(x)|}{\rho^{n+sp}} + \int_{(\mathscr{C}E)\cap \mathscr{C}B_\rho(x)} \frac{dy}{|x-y|^{n+sp}}
$$

$$
= \frac{|E \cap \mathscr{C}B_\rho(x)|}{\rho^{n+sp}} + \int_{(\mathscr{C}E)\cap \mathscr{C}B_\rho(x)} \frac{dy}{|x-y|^{n+sp}}
$$

$$
\geq \int_{E \cap \mathscr{C}B_\rho(x)} \frac{dy}{|x-y|^{n+sp}} + \int_{(\mathscr{C}E)\cap \mathscr{C}B_\rho(x)} \frac{dy}{|x-y|^{n+sp}}
$$

$$
= \int_{\mathscr{C}B_\rho(x)} \frac{dy}{|x-y|^{n+sp}}.
$$

The desired result easily follows by using polar coordinates centered at x. $\square$

Now, we recall a general statement about a useful summability property (see [74, Lemma 5]).

Lemma 6.2. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp < n$. Fix $T > 1$; let $N \in \mathbb{Z}$ and*

(6.1)
$$
a_k \text{ be a bounded, nonnegative, decreasing sequence}
$$
$$
\text{with } a_k = 0 \text{ for any } k \geq N.
$$

Then,

$$
\sum_{k\in\mathbb{Z}} a_k^{(n-sp)/n} T^k \leq C \sum_{\substack{k\in\mathbb{Z} \\ a_k \neq 0}} a_{k+1} a_k^{-sp/n} T^k,
$$

for a suitable constant $C = C(n,p,s,T) > 0$, independent of N.

Proof. By (6.1),

(6.2)   both $\sum_{k\in\mathbb{Z}} a_k^{(n-sp)/n} T^k$ and $\sum_{\substack{k\in\mathbb{Z} \\ a_k \neq 0}} a_{k+1} a_k^{-sp/n} T^k$ are convergent series.

Moreover, since a_k is nonnegative and decreasing, we have that if $a_k = 0$, then $a_{k+1} = 0$. Accordingly,

$$
\sum_{k\in\mathbb{Z}} a_{k+1}^{(n-sp)/n} T^k = \sum_{\substack{k\in\mathbb{Z} \\ a_k \neq 0}} a_{k+1}^{(n-sp)/n} T^k.
$$

Therefore, we may use the Hölder inequality with exponents $\alpha := n/sp$ and $\beta := n/(n-sp)$ by arguing as follows.

$$\frac{1}{T}\sum_{k\in\mathbb{Z}} a_k^{(n-sp)/n}T^k = \sum_{k\in\mathbb{Z}} a_{k+1}^{(n-sp)/n}T^k$$

$$= \sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}} a_{k+1}^{(n-sp)/n}T^k$$

$$= \sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}} \left(a_k^{sp/(n\beta)}T^{k/\alpha}\right)\left(a_{k+1}^{1/\beta}a_k^{-sp/(n\beta)}T^{k/\beta}\right)$$

$$\leq \left(\sum_{k\in\mathbb{Z}}\left(a_k^{sp/(n\beta)}T^{k/\alpha}\right)^\alpha\right)^{1/\alpha}\left(\sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}}\left(a_{k+1}^{1/\beta}a_k^{-sp/(n\beta)}T^{k/\beta}\right)^\beta\right)^{1/\beta}$$

$$\leq \left(\sum_{k\in\mathbb{Z}}a_k^{(n-sp)/n}T^k\right)^{sp/n}\left(\sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}}a_{k+1}a_k^{-sp/n}T^k\right)^{(n-sp)/n}.$$

So, recalling (6.2), we obtain the desired result.   □

We use the above tools to deal with the measure theoretic properties of the level sets of the functions (see [74, Lemma 6]).

Lemma 6.3. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp < n$. Let*

(6.3) $f \in L^\infty(\mathbb{R}^n)$ *be compactly supported.*

For any $k \in \mathbb{Z}$ let

(6.4) $a_k := \left|\{|f| > 2^k\}\right|.$

Then,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \geq C\sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}}a_{k+1}a_k^{-sp/n}2^{pk},$$

for a suitable constant $C = C(n,p,s) > 0$.

Proof. Notice that

$$\left||f(x)| - |f(y)|\right| \leq |f(x) - f(y)|,$$

and so, by possibly replacing f with $|f|$, we may consider the case in which $f \geq 0$.

We define

(6.5) $A_k := \{|f| > 2^k\}.$

We remark that $A_{k+1} \subseteq A_k$, hence

(6.6) $a_{k+1} \leq a_k.$

We define

$$D_k := A_k \setminus A_{k+1} = \{2^k < f \leq 2^{k+1}\} \qquad \text{and} \qquad d_k := |D_k|.$$

Notice that

(6.7) d_k and a_k are bounded and they become zero when k is large enough,

thanks to (6.3). Also, we observe that the D_k's are disjoint, that

$$(6.8) \qquad \bigcup_{\substack{\ell \in \mathbb{Z} \\ \ell \leq k}} D_\ell = \mathscr{C} A_{k+1}$$

and that

$$(6.9) \qquad \bigcup_{\substack{\ell \in \mathbb{Z} \\ \ell \geq k}} D_\ell = A_k.$$

As a consequence of (6.9), we have that

$$(6.10) \qquad a_k = \sum_{\substack{\ell \in \mathbb{Z} \\ \ell \geq k}} d_\ell$$

and so

$$(6.11) \qquad d_k = a_k - \sum_{\substack{\ell \in \mathbb{Z} \\ \ell \geq k+1}} d_\ell.$$

We stress that the series in (6.10) is convergent, due to (6.7), thus so is the series in (6.11). Similarly, we can define the convergent series

$$(6.12) \qquad S := \sum_{\substack{\ell \in \mathbb{Z} \\ a_{\ell-1} \neq 0}} 2^{p\ell} a_{\ell-1}^{-sp/n} d_\ell.$$

We notice that $D_k \subseteq A_k \subseteq A_{k-1}$, hence $a_{i-1}^{-sp/n} d_\ell \leq a_{i-1}^{-sp/n} a_{\ell-1}$. Therefore

$$(6.13) \qquad \begin{aligned} \Big\{ (i,\ell) \in \mathbb{Z} \text{ s.t. } a_{i-1} \neq 0 \text{ and } a_{i-1}^{-sp/n} d_\ell \neq 0 \Big\} \\ \subseteq \Big\{ (i,\ell) \in \mathbb{Z} \text{ s.t. } a_{\ell-1} \neq 0 \Big\}. \end{aligned}$$

We use (6.13) and (6.6) in the following computation:

$$
\begin{aligned}
\sum_{\substack{i\in\mathbb{Z}\\a_{i-1}\neq 0}}\sum_{\substack{\ell\in\mathbb{Z}\\\ell\geq i+1}} 2^{pi}a_{i-1}^{-sp/n}d_\ell \;&=\; \sum_{\substack{i\in\mathbb{Z}\\a_{i-1}\neq 0}}\sum_{\substack{\ell\in\mathbb{Z}\\\ell\geq i+1\\a_{i-1}^{sp/n}d_\ell\neq 0}} 2^{pi}a_{i-1}^{-sp/n}d_\ell\\[2mm]
&\leq\; \sum_{i\in\mathbb{Z}}\sum_{\substack{\ell\in\mathbb{Z}\\\ell\geq i+1\\a_{\ell-1}\neq 0}} 2^{pi}a_{i-1}^{-sp/n}d_\ell\\[2mm]
&=\; \sum_{\substack{\ell\in\mathbb{Z}\\a_{\ell-1}\neq 0}}\sum_{\substack{i\in\mathbb{Z}\\i\leq\ell-1}} 2^{pi}a_{i-1}^{-sp/n}d_\ell\\[2mm]
&\leq\; \sum_{\substack{\ell\in\mathbb{Z}\\a_{\ell-1}\neq 0}}\sum_{\substack{i\in\mathbb{Z}\\i\leq\ell-1}} 2^{pi}a_{\ell-1}^{-sp/n}d_\ell\\[2mm]
(6.14)\qquad\qquad &=\; \sum_{\substack{\ell\in\mathbb{Z}\\a_{\ell-1}\neq 0}}\sum_{k=0}^{+\infty} 2^{p(\ell-1)}2^{-pk}a_{\ell-1}^{-sp/n}d_\ell \;\leq\; S.
\end{aligned}
$$

Now, we fix $i\in\mathbb{Z}$ and $x\in D_i$: then, for any $j\in\mathbb{Z}$ with $j\leq i-2$ and any $y\in D_j$ we have that

$$
|f(x)-f(y)| \;\geq\; 2^i-2^{j+1} \;\geq\; 2^i-2^{i-1} \;=\; 2^{i-1}
$$

and therefore, recalling (6.8),

$$
\begin{aligned}
\sum_{\substack{j\in\mathbb{Z}\\j\leq i-2}}\int_{D_j}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dy \;&\geq\; 2^{p(i-1)}\sum_{\substack{j\in\mathbb{Z}\\j\leq i-2}}\int_{D_j}\frac{dy}{|x-y|^{n+sp}}\\[2mm]
&=\; 2^{p(i-1)}\int_{\mathscr{C}A_{i-1}}\frac{dy}{|x-y|^{n+sp}}.
\end{aligned}
$$

This and Lemma 6.1 imply that, for any $i\in\mathbb{Z}$ and any $x\in D_i$, we have that

$$
\sum_{\substack{j\in\mathbb{Z}\\j\leq i-2}}\int_{D_j}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dy \geq c_o 2^{pi}a_{i-1}^{-sp/n},
$$

for a suitable $c_o>0$.

As a consequence, for any $i\in\mathbb{Z}$,

$$
(6.15)\qquad \sum_{\substack{j\in\mathbb{Z}\\j\leq i-2}}\int_{D_i\times D_j}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \geq c_o 2^{pi}a_{i-1}^{-sp/n}d_i\,.
$$

Therefore, by (6.11), we conclude that, for any $i\in\mathbb{Z}$,

$$
(6.16)\qquad \sum_{\substack{j\in\mathbb{Z}\\j\leq i-2}}\int_{D_i\times D_j}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \geq c_o\left[2^{pi}a_{i-1}^{-sp/n}a_i - \sum_{\substack{\ell\in\mathbb{Z}\\\ell\geq i+1}} 2^{pi}a_{i-1}^{-sp/n}d_\ell\right].
$$

By (6.12) and (6.15), we have that

$$(6.17) \qquad \sum_{\substack{i\in\mathbb{Z} \\ a_{i-1}\neq 0}} \sum_{\substack{j\in\mathbb{Z} \\ j\leq i-2}} \int_{D_i\times D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dx\, dy \geq c_o S.$$

Then, using (6.16), (6.14) and (6.17),

$$\sum_{\substack{i\in\mathbb{Z} \\ a_{i-1}\neq 0}} \sum_{\substack{j\in\mathbb{Z} \\ j\leq i-2}} \int_{D_i\times D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dx\, dy$$

$$\geq c_o \left[\sum_{\substack{i\in\mathbb{Z} \\ a_{i-1}\neq 0}} 2^{pi} a_{i-1}^{-sp/n} a_i - \sum_{\substack{i\in\mathbb{Z} \\ a_{i-1}\neq 0}} \sum_{\substack{\ell\in\mathbb{Z} \\ \ell\geq i+1}} 2^{pi} a_{i-1}^{-sp/n} d_\ell \right]$$

$$\geq c_o \left[\sum_{\substack{i\in\mathbb{Z} \\ a_{i-1}\neq 0}} 2^{pi} a_{i-1}^{-sp/n} a_i - S \right]$$

$$\geq c_o \sum_{\substack{i\in\mathbb{Z} \\ a_{i-1}\neq 0}} 2^{pi} a_{i-1}^{-sp/n} a_i - \sum_{\substack{i\in\mathbb{Z} \\ a_{i-1}\neq 0}} \sum_{\substack{j\in\mathbb{Z} \\ j\leq i-2}} \int_{D_i\times D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dx\, dy.$$

That is, by taking the last term to the left hand side,

$$(6.18) \qquad \sum_{\substack{i\in\mathbb{Z} \\ a_{i-1}\neq 0}} \sum_{\substack{j\in\mathbb{Z} \\ j\leq i-2}} \int_{D_i\times D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dx\, dy \geq c_o \sum_{\substack{i\in\mathbb{Z} \\ a_{i-1}\neq 0}} 2^{pi} a_{i-1}^{-sp/n} a_i,$$

up to relabeling the constant c_0.

On the other hand, by symmetry,

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dx\, dy = \sum_{i,j\in\mathbb{Z}} \int_{D_i\times D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dx\, dy$$

$$= 2 \sum_{\substack{i,j\in\mathbb{Z} \\ j\leq i}} \int_{D_i\times D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dx\, dy$$

$$(6.19) \qquad\qquad \geq 2 \sum_{\substack{i\in\mathbb{Z} \\ a_{i-1}\neq 0}} \sum_{\substack{j\in\mathbb{Z} \\ j\leq i-2}} \int_{D_i\times D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dx\, dy.$$

Then, the desired result plainly follows from (6.18) and (6.19). $\qquad\square$

Lemma 6.4. *Let $q\in[1,\infty)$. Let $f:\mathbb{R}^n\to\mathbb{R}$ be a measurable function. For any $N\in\mathbb{N}$, let*

$$(6.20) \qquad f_N(x) := \max\big\{\min\{f(x),N\},-N\big\} \quad \forall x\in\mathbb{R}^n.$$

Then

$$\lim_{N\to+\infty} \|f_N\|_{L^q(\mathbb{R}^n)} = \|f\|_{L^q(\mathbb{R}^n)}.$$

Proof. We denote by $|f|_N$ the function obtained by cutting $|f|$ at level N. We have that $|f|_N = |f_N|$ and so, by Fatou Lemma, we obtain that

$$\liminf_{N\to+\infty} \|f_N\|_{L^q(\mathbb{R}^n)} = \liminf_{N\to+\infty} \left(\int_{\mathbb{R}^n} |f|_N^q\right)^{\frac{1}{q}} \geq \left(\int_{R^n} |f|^q\right)^{\frac{1}{q}} = \|f\|_{L^q(\mathbb{R}^n)}.$$

The reverse inequality easily follows by the fact that $|f|_N(x) \leq |f(x)|$ for any $x \in \mathbb{R}^n$. □

Taking into account the previous lemmas, we are able to give an elementary proof of the Sobolev-type inequality stated in the following theorem.

Theorem 6.5. *Let $s \in (0,1)$ and $p \in [1, +\infty)$ such that $sp < n$. Then there exists a positive constant $C = C(n,p,s)$ such that, for any measurable and compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$, we have*

$$(6.21) \qquad \|f\|_{L^{p^\star}(\mathbb{R}^n)}^p \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}}\, dx\, dy.$$

where $p^\star = p^\star(n,s)$ is the so-called "fractional critical exponent" and it is equal to $np/(n - sp)$.

Consequently, the space $W^{s,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p, p^\star]$.

Proof. First, we note that if the right hand side of (6.21) is unbounded then the claim in the theorem plainly follows. Thus, we may suppose that f is such that

$$(6.22) \qquad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}}\, dx\, dy \; < \; +\infty.$$

Moreover, we can suppose, without loss of generality, that

$$(6.23) \qquad\qquad\qquad f \in L^\infty(\mathbb{R}^n).$$

Indeed, if (6.22) holds for bounded functions, then it holds also for the function f_N, obtained by any (possibly unbounded) f by cutting at levels $-N$ and $+N$ (see (6.20)). Therefore, by Lemma 6.4 and the fact that (6.22) together with the Dominated Convergence Theorem imply

$$\lim_{N\to+\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_N(x) - f_N(y)|^p}{|x - y|^{n+sp}}\, dx\, dy = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}}\, dx\, dy,$$

we obtain estimate (6.21) for the function f.

Now, take a_k and A_k defined by (6.4) and (6.5), respectively. We have

$$\|f\|_{L^{p^\star}(\mathbb{R}^n)}^{p^\star} = \sum_{k\in\mathbb{Z}} \int_{A_k\setminus A_{k+1}} |f(x)|^{p^\star}\, dx \leq \sum_{k\in\mathbb{Z}} \int_{A_k\setminus A_{k+1}} (2^{k+1})^{p^\star}\, dx$$

$$\leq \sum_{k\in\mathbb{Z}} 2^{(k+1)p^\star} a_k.$$

That is,

$$\|f\|_{L^{p^\star}(\mathbb{R}^n)}^p \leq 2^p \left(\sum_{k\in\mathbb{Z}} 2^{kp^\star} a_k \right)^{p/p^\star}.$$

Thus, since $p/p^\star = (n-sp)/n = 1 - sp/n < 1$,

$$(6.24) \qquad \|f\|_{L^{p^\star}(\mathbb{R}^n)}^p \leq 2^p \sum_{k\in\mathbb{Z}} 2^{kp} a_k^{(n-sp)/n}$$

and, then, by choosing $T = 2^p$, Lemma 6.2 yields

$$(6.25) \qquad \|f\|_{L^{p^\star}(\mathbb{R}^n)}^p \leq C \sum_{\substack{k\in\mathbb{Z} \\ a_k\neq 0}} 2^{kp} a_{k+1} a_k^{-\frac{sp}{k}},$$

for a suitable constant C depending on n, p and s.

Finally, it suffices to apply Lemma 6.3 and we obtain the desired result, up to relabeling the constant C in (6.25).

Furthermore, the embedding for $q \in (p, p^\star)$ follows from standard application of Hölder inequality. $\qquad \square$

Remark 6.6. From Lemma 6.1, it follows that

$$(6.26) \qquad \int_E \int_{\mathscr{C}E} \frac{dx\, dy}{|x-y|^{n+sp}} \geq c(n,s)\, |E|^{(n-sp)/n}$$

for all measurable sets E with finite measure.

On the other hand, we see that (6.21) reduces to (6.26) when $f = \chi_E$, so (6.26) (and thus Lemma 6.1) may be seen as a Sobolev-type inequality for sets.

It is worth to remark that in the case of Ω being a general open set in $\mathbb{R}^n$, the statements in Theorem 6.5 still hold for $W_0^{s,p}(\Omega)$, namely

$$W_0^{s,p}(\Omega) \hookrightarrow L^{p^\star}(\Omega),$$

since one can extend any $f \in W_0^{s,p}(\Omega)$ on all $\mathbb{R}^n$, by setting f equal to zero outside Ω. The above embedding does not generally hold for the space $W^{s,p}(\Omega)$ since it not always possible to extend a function $f \in W^{s,p}(\Omega)$ to a function $\tilde{f} \in W^{s,p}(\mathbb{R}^n)$. In order to be allowed to do that, we should require further regularity assumptions on Ω (see Section 5).

Theorem 6.7. *Let $s \in (0,1)$ and $p \in [1, +\infty)$ such that $sp < n$. Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C = C(n, p, s, \Omega)$ such that, for any $f \in W^{s,p}(\Omega)$, we have*

(6.27) $\|f\|_{L^q(\Omega)} \leq C\|f\|_{W^{s,p}(\Omega)},$

for any $q \in [p, p^\star]$; i.e., the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p, p^\star]$.

If, in addition, Ω is bounded, then the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [1, p^\star]$.

Proof. Let $f \in W^{s,p}(\Omega)$. Since $\Omega \subseteq \mathbb{R}^n$ is an extension domain for $W^{s,p}$, then there exists a constant $C_1 = C_1(n, p, s, \Omega) > 0$ such that

(6.28) $\|\tilde{f}\|_{W^{s,p}(\mathbb{R}^n)} \leq C_1\|f\|_{W^{s,p}(\Omega)},$

with $\tilde{f}$ such that $\tilde{f}(x) = f(x)$ for x a.e. in Ω.

On the other hand, by Theorem 6.5, the space $W^{s,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p, p^\star]$; i.e., there exists a constant $C_2 = C_2(n, p, s) > 0$ such that

(6.29) $\|\tilde{f}\|_{L^q(\mathbb{R}^n)} \leq C_2\|\tilde{f}\|_{W^{s,p}(\mathbb{R}^n)}.$

Combining (6.28) with (6.29), we get

$$\begin{aligned} \|f\|_{L^q(\Omega)} &= \|\tilde{f}\|_{L^q(\Omega)} \leq \|\tilde{f}\|_{L^q(\mathbb{R}^n)} \leq C_2\|\tilde{f}\|_{W^{s,p}(\mathbb{R}^n)} \\ &\leq C_2 C_1 \|f\|_{W^{s,p}(\Omega)}, \end{aligned}$$

that gives the inequality in (6.27), by choosing $C = C_2 C_1$.

In the case of Ω being bounded, the embedding for $q \in [1, p)$ plainly follows from (6.27), by using the Hölder inequality. $\square$

Remark 6.8. A simple scaling argument shows that in the critical case $q = p^\star$ the constant C in Theorem 6.7 does not depend on Ω and then it coincides with the $1/p$-power of the constant C in (6.21).

6.1. **The case $sp = n$.** We note that when $sp \to n$ the critical exponent $p^\star$ goes to ∞ and so it is not surprising that, in this case, if f is in $W^{s,p}$ then f belongs to L^q for any q, as stated in the following two theorems.

Theorem 6.9. *Let $s \in (0,1)$ and $p \in [1, +\infty)$ such that $sp = n$. Then there exists a positive constant $C = C(n, p, s)$ such that, for any measurable and compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$, we have*

(6.30) $\|f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{W^{s,p}(\mathbb{R}^n)},$

for any $q \in [p, \infty)$; i.e., the space $W^{s,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p, \infty)$.

Theorem 6.10. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp = n$. Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C = C(n,p,s,\Omega)$ such that, for any $f \in W^{s,p}(\Omega)$, we have*

$$(6.31) \qquad \|f\|_{L^q(\Omega)} \leq C\|f\|_{W^{s,p}(\Omega)},$$

for any $q \in [p,\infty)$; i.e., the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p,\infty)$.

If, in addition, Ω is bounded, then the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [1,\infty)$.

The proofs can be obtained by simply combining Proposition 2.1 with Theorem 6.5 and Theorem 6.7, respectively.

7. COMPACT EMBEDDINGS

In this section, we state and prove some compactness results involving the fractional spaces $W^{s,p}(\Omega)$ in bounded domains. The main proof is a modification of the one of the classical Riesz-Frechet-Kolmogorov Theorem (see [50, 69]) and, again, it is self-contained and it does not require to use Besov or other interpolation spaces, nor Fourier transform and semigroup flows (see [26, Theorem 1.5]). We refer to [68, Lemma 6.11] for the case $p = q = 2$.

Theorem 7.1. *Let $s \in (0,1)$, $p \in [1,+\infty)$, $q \in [1,p]$, $\Omega \subset \mathbb{R}^n$ be a bounded extension domain for $W^{s,p}$ and $\mathscr{T}$ be a bounded subset of $L^p(\Omega)$. Suppose that*

$$\sup_{f \in \mathscr{T}} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \; < \; +\infty.$$

Then $\mathscr{T}$ is pre-compact in $L^q(\Omega)$.

Proof. We want to show that $\mathscr{T}$ is totally bounded in $L^q(\Omega)$, i.e., for any $\varepsilon \in (0,1)$ there exist $\beta_1, \ldots, \beta_M \in L^q(\Omega)$ such that for any $f \in \mathscr{T}$ there exists $j \in \{1, \ldots, M\}$ such that

$$(7.1) \qquad \|f - \beta_j\|_{L^q(\Omega)} \leq \varepsilon.$$

Since Ω is an extension domain, there exists a function $\tilde{f}$ in $W^{s,p}(\mathbb{R}^n)$ such that $\|\tilde{f}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|f\|_{W^{s,p}(\Omega)}$. Thus, for any cube Q containing Ω, we have

$$\|\tilde{f}\|_{W^{s,p}(Q)} \leq \|\tilde{f}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|f\|_{W^{s,p}(\Omega)}.$$

Observe that, since Q is a bounded open set, $\tilde{f}$ belongs also to $L^q(Q)$ for any $q \in [1,p]$. Now, for any $\varepsilon \in (0,1)$, we let

$$C_o := 1 + \sup_{f \in \mathscr{T}} \|\tilde{f}\|_{L^q(Q)} + \sup_{f \in \mathscr{T}} \int_Q \int_Q \frac{|\tilde{f}(x) - \tilde{f}(y)|^p}{|x-y|^{n+sp}} \, dx \, dy,$$

$$\rho = \rho_\varepsilon := \left(\frac{\varepsilon}{2C_o^{\frac{1}{q}} \, n^{\frac{n+sp}{2p}}} \right)^{\frac{1}{s}} \quad \text{and} \quad \eta = \eta_\varepsilon := \frac{\varepsilon \, \rho^{\frac{n}{q}}}{2},$$

and we take a collection of disjoints cubes $Q_1, \ldots, Q_N$ of side ρ such that

$$\Omega \subseteq Q = \bigcup_{j=1}^{N} Q_j.$$

For any $x \in \Omega$, we define

(7.2) $j(x)$ as the unique integer in $\{1, \ldots, N\}$ for which $x \in Q_{j(x)}$.

Also, for any $f \in \mathscr{T}$, let

$$P(f)(x) := \frac{1}{|Q_{j(x)}|} \int_{Q_{j(x)}} \tilde{f}(y) \, dy.$$

Notice that

$$P(f + g) = P(f) + P(g) \quad \text{for any } f, g \in \mathscr{T}$$

and that $P(f)$ is constant, say equal to $q_j(f)$, in any Q_j, for $j \in \{1, \ldots, N\}$. Therefore, we can define

$$R(f) := \rho^{n/q}\big(q_1(f), \ldots, q_N(f)\big) \in \mathbb{R}^N$$

and consider the spatial q-norm in $\mathbb{R}^N$ as

$$\|v\|_q := \left(\sum_{j=1}^{N} |v_j|^q \right)^{\frac{1}{q}}, \quad \text{for any } v \in \mathbb{R}^N.$$

We observe that $R(f + g) = R(f) + R(g)$. Moreover,

$$\|P(f)\|_{L^q(\Omega)}^q = \sum_{j=1}^{N} \int_{Q_j \cap \Omega} |P(f)(x)|^q \, dx$$

(7.3) $$\leq \rho^n \sum_{j=1}^{N} |q_j(f)|^q = \|R(f)\|_q^q \leq \frac{\|R(f)\|_q^q}{\rho^n}.$$

Also, by Hölder inequality,

$$\|R(f)\|_q^q = \sum_{j=1}^{N} \rho^n |q_j(f)|^q = \frac{1}{\rho^{n(q-1)}} \sum_{j=1}^{N} \left| \int_{Q_j} \tilde{f}(y) \, dy \right|^q$$

$$\leq \sum_{j=1}^{N} \int_{Q_j} |\tilde{f}(y)|^q \, dy = \int_Q |\tilde{f}(y)|^q \, dy = \|\tilde{f}\|_{L^q(Q)}^q.$$

In particular,

$$\sup_{f \in \mathscr{T}} \|R(f)\|_q^q \leq C_o,$$

that is, the set $R(\mathscr{T})$ is bounded in $\mathbb{R}^N$ (with respect to the q-norm of $\mathbb{R}^N$ as well as to any equivalent norm of $\mathbb{R}^N$) and so, since it is finite dimensional, it is totally bounded. Therefore, there exist $b_1, \ldots, b_M \in \mathbb{R}^N$ such that

$$(7.4) \qquad\qquad R(\mathscr{T}) \subseteq \bigcup_{i=1}^{M} B_\eta(b_i),$$

where the balls B_η are taken in the q-norm of $\mathbb{R}^N$.

For any $i \in \{1, \ldots, M\}$, we write the coordinates of b_i as $b_i = (b_{i,1}, \ldots, b_{i,N}) \in \mathbb{R}^N$. For any $x \in \Omega$, we set

$$\beta_i(x) := \rho^{-n/q}\, b_{i,j(x)},$$

where $j(x)$ is as in (7.2).

Notice that β_i is constant on Q_j, i.e. if $x \in Q_j$ then

$$(7.5) \qquad\qquad P(\beta_i)(x) = \rho^{-\frac{n}{q}} b_{i,j} = \beta_i(x)$$

and so $q_j(\beta_i) = \rho^{-\frac{n}{q}} b_{i,j}$; thus

$$(7.6) \qquad\qquad R(\beta_i) = b_i.$$

Furthermore, for any $f \in \mathscr{T}$

$$
\begin{aligned}
\|f - P(f)\|_{L^q(\Omega)}^q \;&=\; \sum_{j=1}^{N} \int_{Q_j \cap \Omega} |f(x) - P(f)(x)|^q \, dx \\[2mm]
&=\; \sum_{j=1}^{N} \int_{Q_j \cap \Omega} \left| f(x) - \frac{1}{|Q_j|} \int_{Q_j} \tilde{f}(y)\, dy \right|^q dx \\[2mm]
&=\; \sum_{j=1}^{N} \int_{Q_j \cap \Omega} \frac{1}{|Q_j|^q} \left| \int_{Q_j} f(x) - \tilde{f}(y)\, dy \right|^q dx \\[2mm]
(7.7) \qquad &\leq\; \frac{1}{\rho^{nq}} \sum_{j=1}^{N} \int_{Q_j \cap \Omega} \left[\int_{Q_j} |f(x) - \tilde{f}(y)|\, dy \right]^q dx.
\end{aligned}
$$

Now for any fixed $j \in 1, \cdots, N$, by Hölder inequality with p and $p/(p-1)$ we get

$$\frac{1}{\rho^{nq}} \left[\int_{Q_j} |f(x) - \tilde{f}(y)| \, dy \right]^q \leq \frac{1}{\rho^{nq}} |Q_j|^{\frac{q(p-1)}{p}} \left[\int_{Q_j} |f(x) - \tilde{f}(y)|^p \, dy \right]^{\frac{q}{p}}$$

$$= \frac{1}{\rho^{nq/p}} \left[\int_{Q_j} |f(x) - \tilde{f}(y)|^p \, dy \right]^{\frac{q}{p}}$$

$$\leq \frac{1}{\rho^{nq/p}} n^{\left(\frac{n+sp}{2}\right)\frac{q}{p}} \rho^{\frac{q}{p}(n+sp)} \left[\int_{Q_j} \frac{|f(x) - \tilde{f}(y)|^p}{|x-y|^{n+sp}} \, dy \right]^{\frac{q}{p}}$$

$$(7.8) \qquad \leq n^{\left(\frac{n+sp}{2}\right)\frac{q}{p}} \rho^{sq} \left[\int_Q \frac{|f(x) - \tilde{f}(y)|^p}{|x-y|^{n+sp}} \, dy \right]^{\frac{q}{p}}.$$

Hence, combining (7.7) with (7.8), we obtain that

$$\|f - P(f)\|_{L^q(\Omega)}^q \leq n^{\left(\frac{n+sp}{2}\right)\frac{q}{p}} \rho^{sq} \int_Q \left[\int_Q \frac{|\tilde{f}(x) - \tilde{f}(y)|^p}{|x-y|^{n+sp}} \, dy \right]^{\frac{q}{p}} dx$$

$$(7.9) \qquad \leq n^{\left(\frac{n+sp}{2}\right)\frac{q}{p}} \rho^{sq} \left[\int_Q \int_Q \frac{|\tilde{f}(x) - \tilde{f}(y)|^p}{|x-y|^{n+sp}} \, dy \, dx \right]^{\frac{q}{p}}$$

$$\leq C_o \, n^{\left(\frac{n+sp}{2}\right)\frac{q}{p}} \rho^{sq} = \frac{\varepsilon^q}{2^q}.$$

where (7.9) follows from Jensen inequality since $t \mapsto |t|^{q/p}$ is a concave function for any fixed p and q such that $q/p \leq 1$.

Consequently, for any $j \in \{1, ..., M\}$, recalling (7.3) and (7.5)

$$\|f - \beta_j\|_{L^q(\Omega)} \leq \|f - P(f)\|_{L^q(\Omega)} + \|P(\beta_j) - \beta_j\|_{L^q(\Omega)} + \|P(f - \beta_j)\|_{L^q(\Omega)}$$

$$(7.10) \qquad \leq \frac{\varepsilon}{2} + \frac{\|R(f) - R(\beta_j)\|_q}{\rho^{n/q}}.$$

Now, given any $f \in \mathscr{T}$, we recall (7.4) and (7.6) and we take $j \in \{1, \dots, M\}$ such that $R(f) \in B_\eta(b_j)$. Then, (7.5) and (7.10) give that

$$(7.11) \qquad \|f - \beta_j\|_{L^q(\Omega)} \leq \frac{\varepsilon}{2} + \frac{\|R(f) - b_j\|_q}{\rho^{n/q}} \leq \frac{\varepsilon}{2} + \frac{\eta}{\rho^{n/q}} = \varepsilon.$$

This proves (7.1), as desired. $\qquad \square$

Corollary 7.2. *Let $s \in (0,1)$ and $p \in [1, +\infty)$ such that $sp < n$. Let $q \in [1, p^\star)$, $\Omega \subseteq \mathbb{R}^n$ be a bounded extension domain for $W^{s,p}$ and $\mathscr{T}$ be a bounded subset of $L^p(\Omega)$.*

Suppose that

$$\sup_{f \in \mathscr{T}} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \; < \; +\infty.$$

Then $\mathscr{T}$ is pre-compact in $L^q(\Omega)$.

Proof. First, note that for $1 \le q \le p$ the compactness follows from Theorem 7.1.

For any $q \in (p, p^\star)$, we may take $\theta = \theta(p, p^\star, q) \in (0, 1)$ such that $1/q = \theta/p + 1 - \theta/p^\star$, thus for any $f \in \mathscr{T}$ and β_j with $j \in \{1, ..., N\}$ as in the theorem above, using Hölder inequality with $p/(\theta q)$ and $p^\star/((1 - \theta)q)$, we get

$$
\begin{aligned}
\|f - \beta_j\|_{L^q(\Omega)} &= \left(\int_\Omega |f - \beta_j|^{q\theta} \, |f - \beta_j|^{q(1-\theta)} \, dx \right)^{1/q} \\[2mm]
&\le \left(\int_\Omega |f - \beta_j|^p \, dx \right)^{\theta/p} \left(\int_\Omega |f - \beta_j|^{p^\star} \, dx \right)^{(1-\theta)/p^\star} \\[2mm]
&= \|f - \beta_j\|_{L^{p^\star}(\Omega)}^{1-\theta} \, \|f - \beta_j\|_{L^p(\Omega)}^{\theta} \\[2mm]
&\le C\|f - \beta_j\|_{W^{s,p}(\Omega)}^{1-\theta} \, \|f - \beta_j\|_{L^p(\Omega)}^{\theta} \le \tilde{C}\varepsilon^\theta,
\end{aligned}
$$

where the last inequalities comes directly from (7.11) and the continuos embedding (see Theorem 6.7). $\qquad\square$

Remark 7.3. As well known in the classical case $s = 1$ (and, more generally, when s is an integer), also in the fractional case the lack of compactness for the critical embedding ($q = p^\star$) is not surprising, because of translation and dilation invariance (see [67] for various results in this direction, for any $0 < s < n/2$).

Notice that the regularity assumption on Ω in Theorem 7.1 and Corollary 7.2 cannot be dropped (see Example 9.2 in Section 9).

8. Hölder regularity

In this section we will show certain regularity properties for functions in $W^{s,p}(\Omega)$ when $sp > n$ and Ω is an extension domain for $W^{s,p}$.

The main result is stated in the forthcoming Theorem 8.2. First, we need a simple technical lemma, whose proof can be found in [42] (for instance).

Lemma 8.1. ([42, Lemma 2.2]). *Let $p \in [1, +\infty)$ and $sp \in (n, n + p]$. Let $\Omega \subset \mathbb{R}^n$ be an extension domain for $W^{s,p}$ and f be a function in $W^{s,p}(\Omega)$. Then, for any $x_0 \in \Omega$ and R, R', with $0 < R' < R < \mathrm{diam}(\Omega)$, we have*

$$(8.1) \qquad |\langle f \rangle_{B_R(x_0) \cap \Omega} - \langle f \rangle_{B_{R'}(x_0) \cap \Omega}| \le c \, [f]_{p,sp} \, |B_R(x_0) \cap \Omega|^{(sp-n)/np}$$

where

$$[f]_{p,sp} := \left(\sup_{\substack{x_0 \in \Omega \\ \rho > 0}} \rho^{-sp} \int_{B_\rho(x_0) \cap \Omega} |f(x) - \langle f \rangle_{B_\rho(x_0) \cap \Omega}|^p \, dx \right)^{\frac{1}{p}}$$

and

$$\langle f \rangle_{B_\rho(x_0) \cap \Omega} := \frac{1}{|B_\rho(x_0) \cap \Omega|} \int_{B_\rho(x_0) \cap \Omega} f(x) dx.$$

Theorem 8.2. *Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,p}$ and let $p \in [1, +\infty)$, $s \in (0,1)$ such that $sp > n$. Then, there exists $C > 0$, only depending on n, p and Ω, such that*

$$(8.2) \qquad \|f\|_{C^{0,\alpha}(\Omega)} \leq C \left(\|f\|_{L^p(\Omega)}^p + \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}},$$

for any $f \in L^p(\Omega)$, with $\alpha := (sp - n)/p$.

Proof. In the following, we will denote by C suitable positive quantities, possibly different from line to line, and possibly depending on p and s.

First, we notice that if the right hand side of (8.2) is not finite, then we are done. Thus, we may suppose that

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \leq C,$$

for some $C > 0$.

Second, since Ω is an extension domain for $W^{s,p}$, we can extend any f to a function $\tilde{f}$ such that $\|\tilde{f}\|_{W^{s,p}(\mathbb{R}^n)} \leq C \|f\|_{W^{s,p}(\Omega)}$.

Now, for any bounded measurable set $U \subset \mathbb{R}^n$, we consider the average value of the function $\tilde{f}$ in U, given by

$$\langle \tilde{f} \rangle_U := \frac{1}{|U|} \int_U \tilde{f}(x) \, dx.$$

For any $\xi \in \mathbb{R}^n$, the Hölder inequality yields

$$\left| \xi - \langle \tilde{f} \rangle_U \right|^p = \frac{1}{|U|^p} \left| \int_U \xi - \tilde{f}(y) \, dy \right|^p \leq \frac{1}{|U|} \int_U |\xi - \tilde{f}(y)|^p \, dy.$$

Accordingly, by taking $x_o \in \Omega$ and $U := B_r(x_o)$, $\xi := \tilde{f}(x)$ and integrating over $B_r(x_o)$, we obtain that

$$\int_{B_r(x_o)} |\tilde{f}(x) - \langle \tilde{f} \rangle_{B_r(x_o)}|^p \, dx$$

$$\leq \frac{1}{|B_r(x_o)|} \int_{B_r(x_o)} \int_{B_r(x_o)} |\tilde{f}(x) - \tilde{f}(y)|^p \, dx \, dy.$$

Hence, since $|x - y| \leq 2r$ for any $x, y \in B_r(x_o)$, we deduce that

$$\int_{B_r(x_o)} |\tilde{f}(x) - \langle \tilde{f} \rangle_{B_r(x_o)}|^p \, dx$$

$$(8.3) \qquad \leq \frac{(2r)^{n+sp}}{|B_r(x_o)|} \int_{B_r(x_o)} \int_{B_r(x_o)} \frac{|\tilde{f}(x) - \tilde{f}(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \leq \frac{2^{n+sp} \, r^{sp} C \|f\|_{W^{s,p}(\Omega)}^p}{|B_1|},$$

that implies

$$(8.4) \qquad\qquad\qquad [f]_{p,sp}^p \leq C \|f\|_{W^{s,p}(\Omega)}^p,$$

for a suitable constant C.

Now, we will show that f is a continuos function. Taking into account (8.1), it follows that the sequence of functions $x \to \langle f \rangle_{B_R(x) \cap \Omega}$ converges uniformly in $x \in \Omega$ when $R \to 0$. In particular the limit function g will be continuos and the same holds for f, since by Lebesgue theorem we have that

$$\lim_{R \to 0} \frac{1}{|B_R(x) \cap \Omega|} \int_{B_R(x) \cap \Omega} f(y) \, dy = f(x) \quad \text{for almost every } x \in \Omega.$$

Now, take any $x, y \in \Omega$ and set $R = |x - y|$. We have

$$|f(x) - f(y)| \leq |f(x) - \langle \tilde{f} \rangle_{B_{2R}(x)}| + |\langle \tilde{f} \rangle_{B_{2R}(x)} - \langle \tilde{f} \rangle_{B_{2R}(y)}| + |\langle \tilde{f} \rangle_{B_{2R}(y)} - f(y)|$$

We can estimate the first and the third term of right hand-side of the above inequality using Lemma 8.1. Indeed, getting the limit in (8.1) as $R' \to 0$ and writing $2R$ instead of R, for any $x \in \Omega$ we get

$$(8.5) \qquad |\langle \tilde{f} \rangle_{B_{2R}(x)} - f(x)| \leq c \, [f]_{p,sp} |B_{2R}(x)|^{(sp-n)/np} \leq C [f]_{p,sp} \, R^{(sp-n)/p}$$

where the constant C is given by $c \, 2^{(sp-n)/p}/|B_1|$.

On the other hand,

$$|\langle \tilde{f} \rangle_{B_{2R}(x)} - \langle \tilde{f} \rangle_{B_{2R}(y)}| \leq |f(z) - \langle \tilde{f} \rangle_{B_{2R}(x)}| + |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(y)}|$$

and so, integrating on $z \in B_{2R}(x) \cap B_{2R}(y)$, we have

$$|B_{2R}(x) \cap B_{2R}(y)| \, |\langle \tilde{f} \rangle_{B_{2R}(x)} - \langle \tilde{f} \rangle_{B_{2R}(y)}|$$

$$\leq \int_{B_{2R}(x) \cap B_{2R}(y)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(x)}| \, dz + \int_{B_{2R}(x) \cap B_{2R}(y)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(y)}| \, dz$$

$$\leq \int_{B_{2R}(x)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(x)}| \, dz + \int_{B_{2R}(y)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(y)}| \, dz$$

Furthermore, since $B_R(x) \cup B_R(y) \subset \big(B_{2R}(x) \cap B_{2R}(y) \big)$, we have

$$|B_R(x)| \leq |B_{2R}(x) \cap B_{2R}(y)| \quad \text{and} \quad |B_R(y)| \leq |B_{2R}(x) \cap B_{2R}(y)|$$

and so

$$|\langle \tilde{f} \rangle_{B_{2R}(x)} - \langle \tilde{f} \rangle_{B_{2R}(y)}| \; \leq \; \frac{1}{|B_R(x)|} \int_{B_{2R}(x)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(x)}| \, dz$$

$$+ \frac{1}{|B_R(y)|} \int_{B_{2R}(y)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(y)}| \, dz.$$

An application of the Hölder inequality gives

$$\frac{1}{|B_R(x)|} \int_{B_{2R}(x)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(x)}| \, dz$$

$$\leq \frac{|B_{2R}(x)|^{(p-1)/p}}{|B_R(x)|} \left(\int_{B_{2R}(x)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(x)}|^p \, dz \right)^{1/p}$$

$$\leq \frac{|B_{2R}(x)|^{(p-1)/p}}{|B_R(x)|} (2R)^s [f]_{p,sp}$$

$$(8.6) \qquad\qquad\qquad \leq C \, [f]_{p,sp} \, R^{(sp-n)/p}.$$

Analogously, we obtain

$$(8.7) \qquad \frac{1}{|B_R(y)|} \int_{B_{2R}(y)} |\tilde{f}(z) - \langle \tilde{f} \rangle_{B_{2R}(y)}| \, dz \leq C \, [f]_{p,sp} \, R^{(sp-n)/p}.$$

Combining (8.5), (8.6) with (8.7) it follows

$$(8.8) \qquad\qquad |f(x) - f(y)| \leq C \, [f]_{p,sp} \, |x - y|^{(sp-n)/p},$$

up to relabeling the constant C.

Therefore, by taking into account (8.4), we can conclude that $f \in C^{0,\alpha}(\Omega)$, with $\alpha = (sp - n)/p$.

Finally, taking $R < \mathrm{diam}(\Omega)$ (note that the latter can be possibly infinity), using estimate in (8.5) and the Hölder inequality we have, for any $x \in \Omega$,

$$|f(x)| \; \leq \; |\langle \tilde{f} \rangle_{B_R(x)}| + |f(x) - \langle \tilde{f} \rangle_{B_R(x)}|$$

$$(8.9) \qquad\qquad \leq \; \frac{C}{|B_R(x)|^{1/p}} \|f\|_{L^p(\Omega)} + c \, [f]_{p,sp} \, |B_R(x)|^\alpha$$

Hence, by (8.4), (8.8) and (8.9), we get

$$\|f\|_{C^{0,\alpha}(\Omega)} \; = \; \|f\|_{L^\infty(\Omega)} + \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

$$\leq \; C \left(\|f\|_{L^p(\Omega)} + [f]_{p,sp} \right)$$

$$\leq \; C \|f\|_{W^{s,p}(\Omega)}.$$

for a suitable positive constant C. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 8.3. The estimate in (8.3) says that f belongs to the Campanato space $\mathscr{L}^{p,\lambda}$, with $\lambda := sp$, (see [20] and, e.g., [42, Definition 2.4]). Then, the conclusion in the proof of Theorem 8.2 is actually an application of the Campanato Isomorphism (see, for instance, [42, Theorem 2.9]).

Just for a matter of curiosity, we observe that, according to the definition (2.1), the fractional Sobolev space $W^{s,\infty}(\Omega)$ could be view as the space of functions

$$\left\{ u \in L^\infty(\Omega) \ : \ \frac{|u(x) - u(y)|}{|x - y|^s} \in L^\infty(\Omega \times \Omega) \right\},$$

but this space just boils down to $C^{0,s}(\Omega)$, that is consistent with the Hölder embedding proved in this section; i.e., taking formally $p = \infty$ in Theorem 8.2, the function u belongs to $C^{0,s}(\Omega)$.

9. SOME COUNTEREXAMPLES IN NON-LIPSCHITZ DOMAINS

When the domain Ω is not Lipschitz, some interesting things happen, as next examples show.

Example 9.1. *Let $s \in (0,1)$. We will construct a function u in $W^{1,p}(\Omega)$ that does not belong to $W^{s,p}(\Omega)$, providing a counterexample to Proposition 2.2 when the domain is not Lipschitz.*

Take any

(9.1)
$$p \in (1/s, +\infty).$$

Due to (9.1), we can fix

(9.2)
$$\kappa > \frac{p+1}{sp-1}.$$

We remark that $\kappa > 1$.

Let us consider the cusp in the plane

$$\mathcal{C} := \big\{(x_1, x_2) \text{ with } x_1 \leq 0 \text{ and } |x_2| \leq |x_1|^\kappa\big\}$$

and take polar coordinates on $\mathbb{R}^2 \backslash \mathcal{C}$, say $\rho = \rho(x) \in (0, +\infty)$ and $\theta = \theta(x) \in (-\pi, \pi)$, with $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \mathcal{C}$.

We define the function $u(x) := \rho(x)\theta(x)$ and the heart-shaped domain $\Omega := (\mathbb{R}^2 \setminus \mathcal{C}) \cap B_1$, with B_1 being the unit ball centered in the origin. Then, $u \in W^{1,p}(\Omega) \setminus W^{s,p}(\Omega)$.

To check this, we observe that

$$\partial_{x_1}\rho \ = \ (2\rho)^{-1}\partial_{x_1}\rho^2 \ = \ (2\rho)^{-1}\partial_{x_1}(x_1^2 + x_2^2) \ = \ \frac{x_1}{\rho}$$

and, in the same way,

$$\partial_{x_2}\rho = \frac{x_2}{\rho}.$$

Accordingly,

$$1 \;=\; \partial_{x_1} x_1 \;=\; \partial_{x_1}(\rho \cos \theta) \;=\; \partial_{x_1}\rho \cos \theta - \rho \sin \theta \partial_{x_1}\theta \;=\; \frac{x_1^2}{\rho^2} - x_2 \partial_{x_1}\theta$$

$$=\; 1 - \frac{x_2^2}{\rho^2} - x_2 \partial_{x_1}\theta.$$

That is

$$\partial_{x_1}\theta = -\frac{x_2}{\rho^2}.$$

By exchanging the roles of x_1 and x_2 (with some care on the sign of the derivatives of the trigonometric functions), one also obtains

$$\partial_{x_2}\theta = \frac{x_1}{\rho^2}.$$

Therefore,

$$\partial_{x_1} u \;=\; \rho^{-1}(x_1 \theta - x_2) \quad \text{and} \quad \partial_{x_2} u \;=\; \rho^{-1}(x_2 \theta + x_1)$$

and so

$$|\nabla u|^2 = \theta^2 + 1 \;\le\; \pi^2 + 1.$$

This shows that $u \in W^{1,p}(\Omega)$.

On the other hand, let us fix $r \in (0,1)$, to be taken arbitrarily small at the end, and let us define $r_0 := r$ and, for any $j \in \mathbb{N}$, $r_{j+1} := r_j - r_j^\kappa$. By induction, one sees that r_j is strictly decreasing, that $r_j > 0$ and so $r_j \in (0,r) \subset (0,1)$. Accordingly, we can define

$$\ell := \lim_{j \to +\infty} r_j \in [0,1].$$

By construction

$$\ell \;=\; \lim_{j \to +\infty} r_{j+1} \;=\; \lim_{j \to +\infty} r_j - r_j^\kappa \;=\; \ell - \ell^\kappa,$$

hence $\ell = 0$. As a consequence,

$$\sum_{j=0}^{+\infty} r_j^\kappa \;=\; \lim_{N \to +\infty} \sum_{j=0}^{N} r_j^\kappa \;=\; \lim_{N \to +\infty} \sum_{j=0}^{N} r_j - r_{j+1}$$

$$\text{(9.3)} \qquad\qquad =\; \lim_{N \to +\infty} r_0 - r_{N+1} \;=\; r.$$

We define

$$\mathcal{D}_j \;:=\; \big\{ (x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \text{ s.t. } x_1, y_1 \in (-r_j, -r_{j+1}),$$
$$x_2 \in (|x_1|^\kappa, 2|x_1|^\kappa) \text{ and } -y_2 \in (|y_1|^\kappa, 2|y_1|^\kappa) \big\}.$$

We observe that

$$\Omega \times \Omega \;\supseteq\; \big\{ (x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \text{ s.t. } x_1, y_1 \in (-r, 0),$$
$$x_2 \in (|x_1|^\kappa, 2|x_1|^\kappa) \text{ and } -y_2 \in (|y_1|^\kappa, 2|y_1|^\kappa) \big\}$$

$$\supseteq\; \bigcup_{j=0}^{+\infty} \mathcal{D}_j,$$

and the union is disjoint. Also,
$$r_{j+1} = r_j(1 - r_j^{\kappa-1}) \geq r_j(1 - r^{\kappa-1}) \geq \frac{r_j}{2},$$
for small r. Hence, if $(x,y) \in \mathcal{D}_j$,
$$|x_1| \leq r_j \leq 2r_{j+1} \leq 2|y_1|$$
and, analogously,
$$|y_1| \leq 2|x_1|.$$
Moreover, if $(x,y) \in \mathcal{D}_j$,
$$|x_1 - y_1| \leq r_j - r_{j+1} = r_j^\kappa \leq 2^\kappa r_{j+1}^\kappa \leq 2^\kappa |x_1|^\kappa$$
and
$$|x_2 - y_2| \leq |x_2| + |y_2| \leq 2|x_1|^\kappa + |y_1|^\kappa \leq 2^{\kappa+1}|x_1|^\kappa.$$
As a consequence, if $(x,y) \in \mathcal{D}_j$,
$$|x - y| \leq 2^{\kappa+2}|x_1|^\kappa.$$
Notice also that, when $(x,y) \in \mathcal{D}_j$, we have $\theta(x) \geq \pi/2$ and $\theta(y) \leq -\pi/2$, so
$$u(x) - u(y) \geq u(x) \geq \frac{\pi \rho(x)}{2} \geq \frac{\pi |x_1|}{2}.$$
As a consequence, for any $(x,y) \in \mathcal{D}_j$,
$$\frac{|u(x) - u(y)|^p}{|x - y|^{2+sp}} \geq c|x_1|^{p-\kappa(2+sp)},$$
for some $c > 0$. Therefore,
$$\iint_{\mathcal{D}_j} \frac{|u(x) - u(y)|^p}{|x - y|^{2+sp}} \, dx \, dy \geq \iint_{\mathcal{D}_j} c|x_1|^{p-\kappa(2+sp)} \, dx \, dy$$
$$= c \int_{-r_j}^{-r_{j+1}} dx_1 \int_{-r_j}^{-r_{j+1}} dy_1 \int_{|x_1|^\kappa}^{2|x_1|^\kappa} dx_2 \int_{-2|y_1|^\kappa}^{-|y_1|^\kappa} dy_2 |x_1|^{p-\kappa(2+sp)}$$
$$= c \int_{-r_j}^{-r_{j+1}} dx_1 \int_{-r_j}^{-r_{j+1}} dy_1 |x_1|^{p-\kappa(2+sp)}|x_1|^\kappa |y_1|^\kappa$$
$$\geq c\,2^{-\kappa} \int_{-r_j}^{-r_{j+1}} dx_1 \int_{-r_j}^{-r_{j+1}} dy_1 |x_1|^{p-\kappa sp}$$
$$\geq c\,2^{-\kappa} \int_{-r_j}^{-r_{j+1}} dx_1 \int_{-r_j}^{-r_{j+1}} dy_1 r_j^{p-\kappa sp}$$
$$= c\,2^{-\kappa} r_j^{p-\kappa sp+2\kappa} = c\,2^{-\kappa} r_j^{\kappa-\alpha},$$
with
(9.4) $$\alpha := \kappa(sp - 1) - p > 1,$$
thanks to (9.2).

In particular,

$$\iint_{\mathcal{D}_j} \frac{|u(x) - u(y)|^p}{|x - y|^{2+sp}} \, dx \, dy \geq c \, 2^{-\kappa} r^{-\alpha} r_j^\kappa$$

and so, by summing up and exploiting (9.3),

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{2+sp}} \, dx \, dy \geq \sum_{j=0}^{+\infty} \iint_{\mathcal{D}_j} \frac{|u(x) - u(y)|^p}{|x - y|^{2+sp}} \, dx \, dy \geq c \, 2^{-\kappa} r^{1-\alpha}.$$

By taking r as small as we wish and recalling (9.4), we obtain that

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{2+sp}} \, dx \, dy = +\infty,$$

so $u \notin W^{s,p}(\Omega)$. $\qquad\square$

Example 9.2. Let $s \in (0, 1)$. We will construct a sequence of functions $\{f_n\}$ bounded in $W^{s,p}(\Omega)$ that does not admit any convergent subsequence in $L^q(\Omega)$, providing a counterexample to Theorem 7.1 when the domain is not Lipschitz.

We follow an observation by [72]. For the sake of simplicity, fix $n = p = q = 2$. We take $a_k := 1/C^k$ for a constant $C > 10$ and we consider the set $\Omega = \bigcup_{k=1}^\infty B_k$ where, for any $k \in \mathbb{N}$, B_k denotes the ball of radius a_k^2 centered in a_k. Notice that

$$a_k \to 0 \text{ as } k \to \infty \quad \text{and} \quad a_k - a_k^2 > a_{k+1} + a_{k+1}^2.$$

Thus, Ω is the union of disjoint balls, it is bounded and it is not a Lipschitz domain.

For any $n \in \mathbb{N}$, we define the function $f_n : \Omega \to \mathbb{R}$ as follows

$$f_n(x) = \begin{cases} \pi^{-\frac{1}{2}} a_n^{-2} & x \in B_n, \\ 0 & x \in \Omega \setminus B_n. \end{cases}$$

We observe that we cannot extract any subsequence convergent in $L^2(\Omega)$ from the sequence of functions $\{f_n\}$, because $f_n(x) \to 0$ as $n \to +\infty$, for any fixed $x \in \Omega$ but

$$\|f_n\|_{L^2(\Omega)}^2 = \int_\Omega |f_n(x)|^2 \, dx = \int_{B_n} \pi^{-1} a_n^{-4} \, dx = 1.$$

Now, we compute the H^s norm of f_n in Ω. We have

$$\int_\Omega \int_\Omega \frac{|f_n(x) - f_n(y)|^2}{|x - y|^{2+2s}} \, dx \, dy = 2 \int_{\Omega \setminus B_n} \int_{B_n} \frac{\pi^{-1} a_n^{-4}}{|x - y|^{2+2s}} \, dx \, dy$$

(9.5)
$$= 2\pi^{-1} \sum_{k \neq n} \int_{B_k} \int_{B_n} \frac{a_n^{-4}}{|x - y|^{2+2s}} \, dx \, dy.$$

Thanks to the choice of $\{a_k\}$ we have that

$$|a_n^2 + a_k^2| = a_n^2 + a_k^2 \leq \frac{|a_n - a_k|}{2}.$$

Thus, since $x \in B_n$, $y \in B_k$, it follows

$$
\begin{aligned}
|x - y| &\geq |a_n - a_n^2 - (a_k + a_k^2)| = |a_n - a_k - (a_n^2 + a_k^2)| \\
&\geq |a_n - a_k| - |a_n^2 + a_k^2| \geq |a_n - a_k| - \frac{|a_n - a_k|}{2} \\
&= \frac{|a_n - a_k|}{2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_{B_k} \int_{B_n} \frac{a_n^{-4}}{|x - y|^{2+2s}} \, dx \, dy &\leq 2^{2+2s} \int_{B_k} \int_{B_n} \frac{a_n^{-4}}{|a_n - a_k|^{2+2s}} \, dx \, dy \\
&= 2^{2+2s} \pi^2 \frac{a_k^4}{|a_n - a_k|^{2+2s}}.
\end{aligned}
$$
(9.6)

Also, if $m \geq j + 1$ we have

$$
(9.7) \qquad a_j - a_m \geq a_j - a_{j+1} = \frac{1}{C^j} - \frac{1}{C^{j+1}} = \frac{1}{C^j}\left(1 - \frac{1}{C}\right) \geq \frac{a_j}{2}
$$

Therefore, combining (9.7) with (9.5) and (9.6), we get

$$
\begin{aligned}
\int_\Omega \int_\Omega \frac{|f_n(x) - f_n(y)|^2}{|x - y|^{2+2s}} \, dx \, dy &\leq 2^{3+2s} \pi \sum_{k \neq n} \frac{a_k^4}{|a_n - a_k|^{2+2s}} \\
&= 2^{3+2s} \pi \left(\sum_{k<n} \frac{a_k^4}{(a_k - a_n)^{2+2s}} + \sum_{k>n} \frac{a_k^4}{(a_n - a_k)^{2+2s}} \right) \\
&\leq 2^{5+4s} \pi \left(\sum_{k<n} \frac{a_k^4}{a_k^{2+2s}} + \sum_{k>n} \frac{a_k^4}{a_n^{2+2s}} \right) \\
&\leq 2^{6+4s} \pi \sum_{k \neq n} a_k^{2-2s} \\
&= 2^{6+4s} \pi \sum_{k \neq n} \left(\frac{1}{C^{2-2s}} \right)^k < +\infty.
\end{aligned}
$$

This shows that $\{f_n\}$ is bounded in $H^s(\Omega)$. $\qquad \square$

REFERENCES

[1] R. A. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] G. ALBERTI, G. BOUCHITTÉ AND P. SEPPECHER, Phase transition with the line-tension effect, *Arch. Rational Mech. Anal.* **144** (1998), no. 1, 1–46.

[3] N. ARONSZAJN, Boundary values of functions with finite Dirichlet integral, *Techn. Report of Univ. of Kansas* **14** (1955), 77–94.

[4] P. W. BATES, On some nonlocal evolution equations arising in materials science. In *Nonlinear dynamics and evolution equations*, Vol. 48 of *Fields Inst. Commun.*, pages 13–52. Amer. Math. Soc., Providence, RI, 2006.

[5] W. BECKNER, Inequalities in Fourier Analysis in $\mathbb{R}^n$, *Proc. Nat. Acad. Sci. USA* **72** (1975), no. 2, 638–641.

[6] O. V. BESOV, On a certain family of functional spaces. Imbedding and continuation theorems, *Dokl. Akad. Nauk SSSR* **126** (1959), 1163–1165.

[7] O. V. BESOV, On some conditions of membership in L^p for derivatives of periodic functions, *Nauĉn. Dokl. Vyss. Skoly. Fiz.-Mat. Nauki* (1959), 13–17.

[8] P. BILER, G. KARCH AND W. A. WOYCZYŃSKI, Critical nonlinearity exponent and self-similar asymptotics for Lévy conservation laws, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **18** (2001), no. 5, 613–637.

[9] H. BREZIS, *How to recognize constant functions. Connections with Sobolev spaces*, Uspekhi Mat. Nauk **57** (2002), no. 4 (346), 59–74; translation in Russian Math. Surveys **57** (2002), no. 4, 693–708.

[10] H. BREZIS AND P. MIRONESCU, *Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces*, J. Evol. Equ. **1** (2001), no. 4, 387–404.

[11] X. CABRÉ AND E. CINTI, Energy estimates and 1-D symmetry for nonlinear equations involving the half-Laplacian, *Discrete and Continuous Dynamical Systems* **28** (2010), 1179-1206.

[12] X. CABRÉ AND J. SOLÀ-MORALES, Layer solutions in a half-space for boundary reactions, *Comm. Pure Appl. Math.* **58** (2005), no. 12, 1678–1732.

[13] L. CAFFARELLI, A. MELLET AND Y. SIRE, Traveling waves for a boundary reaction-diffusion equation, *preprint*, available online at http://arxiv.org/abs/1101.4381v1.

[14] L. CAFFARELLI, J.-M. ROQUEJOFFRE AND O. SAVIN, Non-local minimal surfaces, *Comm. Pure Appl. Math.* **63** (2010), 1111–1144.

[15] L. CAFFARELLI, J.-M. ROQUEJOFFRE AND Y. SIRE, Variational problems with free boundaries for the fractional laplacian, *J. Eur. Math. Soc.* **12** (2010), 1151–1179.

[16] L. CAFFARELLI, S. SALSA AND L. SILVESTRE, Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian, *Invent. Math.* **171** (2008), no. 2, 425–461.

[17] L. CAFFARELLI AND L. SILVESTRE, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32** (2007), 1245–1260.

[18] L. CAFFARELLI AND E. VALDINOCI, Uniform estimates and limiting arguments for nonlocal minimal surfaces, to appear in *Calc. Var. Partial Differential Equations*, available online at http://www.springerlink.com/content/700221730mk71697.

[19] L. CAFFARELLI AND A. VASSEUR, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation *Annals of Math.* **171** (2010), No. 3, 1903–1930.

[20] S. CAMPANATO, Proprietà di hölderianità di alcune classi di funzioni, *Ann. Scuola Norm. Sup. Pisa* **17** (1963), no. 3, 175–188.

[21] M. CHERMISI AND E. VALDINOCI, Fibered nonlinearities for $p(x)$-Laplace equations, *Adv. Calc. Var.* **2**(2009), no. 2, 185–205.

[22] M. CHERMISI AND E. VALDINOCI, A symmetry result for a general class of divergence form PDEs in fibered media. *Nonlinear Anal.* **73**(2010), no. 3, 695–703.

[23] D. COLTON AND R. KRESS, *Inverse acoustic and electromagnetic scattering theory*, Vol. 93 of *Applied Mathematical Sciences*, Springer-Verlag, Berlin, 2nd edition, 1998.

[24] R. CONT AND P. TANKOV, *Financial modelling with jump processes*, Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[25] D. CORDOBA, Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation, *Ann. of Math. (2)* **148** (1998), no. 3, 1135–1152.

[26] A. COTSIOLIS AND N. K. TAVOULARIS, Best constants for Sobolev inequalities for higher order fractional derivatives, *J. Math. Anal. Appl.*, **295** (2004), 225–236.

[27] W. CRAIG AND M. D. GROVES, Hamiltonian long-wave approximations to the water-wave problem, *Wave Motion* **19** (1994), no. 4, 367–389.

[28] W. CRAIG AND D. P. NICHOLLS, Travelling two and three dimensional capillary gravity water waves, *SIAM J. Math. Anal.* **32** (2000), no. 2, 323–359 (electronic).

[29] W. CRAIG, U. SCHANZ AND C. SULEM, The modulational regime of three-dimensional water waves and the Davey-Stewartson system, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14** (1997), no. 5, 615–667.

[30] W. CRAIG, C. SULEM, AND P.-L. SULEM, Nonlinear modulation of gravity waves: a rigorous approach, *Nonlinearity* **5** (1992), no. 2, 497–522.

[31] W. CRAIG AND P. A. WORFOLK, An integrable normal form for water waves in infinite depth, *Phys. D* **84** (1995), no. 3-4, 513–531.

[32] R. DE LA LLAVE AND P. PANAYOTAROS, Gravity waves on the surface of the sphere, *J. Nonlinear Sci.* **6** (1986), no. 2, 147–167.

[33] R. DE LA LLAVE AND E. VALDINOCI, Symmetry for a Dirichlet-Neumann problem arising in water waves, *Math. Res. Lett.* **16** (2009), no. 5, 909918.

[34] J. J. DUISTERMAAT AND V. W. GUILLEMIN, The spectrum of positive elliptic operators and periodic bicharacteristics, *Invent. Math.* **29** (1975), no. 1, 39–79.

[35] G. DUVAUT AND J.-L. LIONS, *Inequalities in mechanics and physics*, Springer-Verlag, Berlin, 1976. Translated from the French by C. W. John, Grundlehren der Mathematischen Wissenschaften, 219.

[36] A. FARINA AND E. VALDINOCI, Rigidity results for elliptic PDEs with uniform limits: an abstract framework with applications, to appear in *Indiana Univ. Math. J.*, available online at http://www.iumj.indiana.edu/IUMJ/Preprints/4433.pdf.

[37] C. FEFFERMAN AND R. DE LA LLAVE, Relativistic stability of matter. I, *Rev. Mat. Iberoamericana* **2** (1986), no. 1-2, 119–213.

[38] G. K. GÄCHTER AND M. J. GROTE, Dirichlet-to-Neumann map for three-dimensional elastic waves, *Wave Motion* **37** (2003), no. 3, 293–311.

[39] E. GAGLIARDO, Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in *n* variabili, *Rend. Sem. Mat. Univ. Padova* **27** (1957), 284–305.

[40] E. GAGLIARDO, Proprietà di alcune classi di funzioni in più variabili, *Ricerche Mat.* **7** (1958), 102–137.

[41] A. GARRONI AND G. PALATUCCI, A singular perturbation result with a fractional norm, in *Variational problems in material science*, *Progress in Nonlinear Differential Equations and Their Applications* **68** (2006), Birkhäuser, Basel, 111–126.

[42] E. GIUSTI, *Metodi diretti nel calcolo delle variazioni*, Unione Matematica Italiana, Bologna, 1994.

[43] M. D. M. GONZALEZ AND R. MONNEAU, Slow motion of particle systems as a limit of a reaction-diffusion equation with half-Laplacian in dimension one, *preprint*, available online at http://arxiv.org/abs/1007.0740v1.

[44] R. Gorenflo and F. Mainardi, Random walk models for Space-fractional diffusion process, *Fractional Calculus and Applied Analysis* **1** (1998), 167–191.

[45] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 2001.

[46] M. J. Grote and C. Kirsch, Dirichlet-to-Neumann boundary conditions for multiple scattering problems, *J. Comput. Phys.* **201** (2004), no. 2, 630–650.

[47] B. Hu and D. P. Nicholls, Analyticity of Dirichlet-Neumann operators on Hölder and Lipschitz domains, *SIAM J. Math. Anal.* **37** (2005), no. 1, 302–320 (electronic).

[48] R. Johnson, Book Review: Theory of function spaces. Bull. Amer. Math. Soc. **13** (1985), no. 1, 76–80.

[49] K. H. Karlsen, F. Petitta and S. Ulusoy, A duality approach to the fractional laplacian with measure data, *Publ. Mat.* **55** (2011), 151–161.

[50] A. N. Kolmogorov, Über Kompaktheit der Funktionenmengen bei der Konvergenz im Mittel, *Nachr. Ges. Wiss. Göttingen* **9** (1931), 60–63.

[51] M. Kurzke, A nonlocal singular perturbation problem with periodic well potential, *ESAIM Control Optim. Calc. Var.* **12** (2006), no. 1, 52–63 (electronic).

[52] P. W. Jones, Quasiconformal mappings and extendability of functions in Sobolev spaces, *Acta Math.* **147**(1981), no. 1-2, 71–88.

[53] N. S. Landkof, *Foundations of Modern Potential Theory*. Translated from Russian, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **180**. Springer-Verlag, Berlin, 1973.

[54] G. Leoni, *A first course in Sobolev spaces*. Graduate Studies in Mathematics **105**, American Mathematical Society, Providence, RI, 2009.

[55] J. L. Lions, Théorèmes de trace et d'interpolation (IV), *Math. Ann.* **151** (1963), 42–56.

[56] J. L. Lions and E. Magenes, *Problémes aux limites non homogénes et applications*, xx+372 pp., Vol. 1. Travaux et Recherches Mathématiques, No. 17, Dunod, Paris, 1968.

[57] P. I. Lizorkin, Boundary properties of functions from "weight" classes, *Dokl. Akad. Nauk SSSR* **132** (1960), 514–517. English translation in *Soviet Math. Dokl.* **1** (1960), 589–593.

[58] A. J. Majda and E. G. Tabak, A two-dimensional model for quasigeostrophic flow: comparison with the two-dimensional Euler flow, *Phys. D* **98** (1996), no. 2-4, 515–522. Nonlinear phenomena in ocean dynamics (Los Alamos, NM, 1995).

[59] V. Maz'ya and T. Shaposhnikova, *Theory of Sobolev multipliers. With applications to differential and integral operators*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **337**. Springer-Verlag, Berlin, 2009.

[60] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.* **339** (2000), 1–77.

[61] E. Milakis and L. Silvestre, Regularity for the nonlinear Signorini problem, *Advances in Math.* **217** (2008), 1301–1312.

[62] G. Mingione, The Calderón-Zygmund theory for elliptic problems with measure data, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)* **6** (2007), 195–261.

[63] G. Mingione, Gradient potential estimates, *J. Eur. Math. Soc.* **13** (2011), 459–486.

[64] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.*, 165 (1972), 207–226.

[65] P. I. NAUMKIN AND I. A. SHISHMARËV, *Nonlinear nonlocal equations in the theory of waves.* American Mathematical Society, Providence, RI, 1994. Translated from the Russian manuscript by Boris Gommerstadt.

[66] D. P. NICHOLLS AND M. TABER, Joint analyticity and analytic continuation of Dirichlet-Neumann operators on doubly perturbed domains, *J. Math. Fluid Mech.* **10** (2008), no. 2, 238–271.

[67] G. PALATUCCI AND A. PISANTE, Sobolev embeddings and concentration-compactness alternative for fractional Sobolev spaces, *submitted paper*, available online at http://mipa.unimes.fr/preprints.html.

[68] G. PALATUCCI, O. SAVIN AND E. VALDINOCI, Local and global minimizers for a variational energy involving a fractional norm, *submitted paper*, available online at http://arxiv.org/abs/1104.1725.

[69] M. RIESZ, Sur les ensembles compacts de fonctions sommables, *Acta Szeged Sect. Math.* **6** (1933), 136–142.

[70] T. RUNST AND W. SICKEL, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations.* de Gruyter Series in Nonlinear Analysis and Applications, no. 3. Walter de Gruyter & Co., Berlin, 1996.

[71] J. J. STOKER, *Water waves: The mathematical theory with applications.* Pure and Applied Mathematics, Vol. IV, Interscience Publishers, Inc., New York, 1957.

[72] O. SAVIN, Personal communication, 2011.

[73] O. SAVIN AND E. VALDINOCI, Elliptic PDEs with fibered nonlinearities, *J. Geom. Anal.* **19** (2009), no. 2, 420–432.

[74] O. SAVIN AND E. VALDINOCI, Density estimates for a nonlocal variational model via the Sobolev inequality, *submitted paper*, available online at http://arxiv.org/abs/1103.6205.

[75] O. SAVIN AND E. VALDINOCI, Density estimates for a variational model driven by the Gagliardo norm, *submitted paper*, available online at http://arxiv.org/abs/1007.2114v3.

[76] L. SILVESTRE, *Regularity of the obstacle problem for a fractional power of the Laplace operator.* Ph.D. Thesis, Austin University, 2005, available online at http://www.math.uchicago.edu/~luis/preprints/luisdissreadable.pdf.

[77] L. SILVESTRE, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Communications on Pure and Applied Mathematics* **60** (2007), no. 1, 67–112.

[78] Y. SIRE AND E. VALDINOCI, Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result, *J. Funct. Anal.* **256** (2009), no. 6, 1842–1864.

[79] L. N. SLOBODECKIJ, Generalized Sobolev spaces and their applications to boundary value problems of partial differential equations, *Leningrad. Gos. Ped. Inst. Učep. Zap.* **197** (1958), 54–112.

[80] E. M. STEIN, *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series **30**, Princeton University Press, Princeton, N.J., 1970.

[81] L. TARTAR, *An Introduction to Sobolev Spaces and Interpolation Spaces.* Lecture Notes of the Unione Matematica Italiana **3**, Springer-Verlag, Berlin Heidelberg 2007.

[82] J. F. TOLAND, The Peierls-Nabarro and Benjamin-Ono equations, *J. Funct. Anal.* **145** (1997), no. 1, 136–150.

[83] H. TRIEBEL, *Interpolation theory, Fourier analysis and function spaces.* Teubner-Texte Math., Teubner, Leipzig, 1977.

[84] H. Triebel, *Interpolation theory, Spaces of Besov-Hardy-Sobolev type*. Teubner-Texte Math., Teubner, Leipzig, 1978.

[85] S. V. Uspenskii, An imbedding theorem for S. L. Sobolev's classes W_p^r of fractional order, *Dokl. Akad. Nauk SSSR* **130** (1960), 992-993. English translation in *Soviet Math. Dokl.* **1** (1960), 132-133.

[86] S. V. Uspenskii, Properties of the classes W_p^r with fractional derivatives on differentiable manifolds, *Dokl. Akad. Nauk SSSR* **132** (1960), 60–62. English translation in *Soviet Math. Dokl.* **1** (1960), 495–497.

[87] E. Valdinoci, From the lung jump random walk to the fractional Laplacian, *Bol. Soc. Esp. Mat. Apl. SeMA* **49** (2009), 33–44.

[88] G. B. Whitham, *Linear and nonlinear waves*. Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1974.

[89] V.E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, *Zh. Prikl. Mekh. Tekh. Fiz.* **9** (1968), 86–94.

[90] W. P. Ziemer, *Weakly Differentiable Functions: Sobolev Spaces and Functions of Bounded Variation*. Springer-Verlag, Berlin, 1989.

E-mail address, Eleonora di Nezza: dinezza@mat.uniroma2.it

E-mail address, Giampiero Palatucci: giampiero.palatucci@unimes.fr

E-mail address, Enrico Valdinoci: enrico@math.utexas.edu

Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 1, 00133 Roma, Italia