FRACTIONAL POROUS MEDIA EQUATIONS: EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS WITH MEASURE DATA

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Abstract. We prove existence and uniqueness of solutions to a class of porous media equations driven by the fractional Laplacian when the initial data are positive finite Radon measures on the Euclidean space $\mathbb{R}^d$. For given solutions without a prescribed initial condition, the problem of existence and uniqueness of the initial trace is also addressed. By the same methods we can also treat weighted fractional porous media equations, with a weight that can be singular at the origin, and must have a sufficiently slow decay at infinity (power-like). In particular, we show that the Barenblatt-type solutions exist and are unique. Such a result has a crucial role in [24], where the asymptotic behavior of solutions is investigated. Our uniqueness result solves a problem left open, even in the non-weighted case, in [42].

1. Introduction

The main goal of this note is to prove existence and uniqueness of solutions to the following problem:

$$\begin{cases} \rho(x)u_t + (-\Delta)^s\,(u^m) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+\,, \\ \rho(x)u = \mu & \text{on } \mathbb{R}^d \times \{0\}\,, \end{cases} \tag{1.1}$$

where we assume that $s \in (0,1)$, $d > 2s$, $m > 1$, μ is a positive finite Radon measure on $\mathbb{R}^d$ (so that $u \geq 0$) and that the (Lebesgue) measurable weight ρ satisfies

$$c|x|^{-\gamma_0} \leq \rho(x) \leq C|x|^{-\gamma_0} \quad \text{a.e. in } B_1 \quad \text{and} \quad c|x|^{-\gamma} \leq \rho(x) \leq C|x|^{-\gamma} \quad \text{a.e. in } B_1^c \tag{1.2}$$

for some $\gamma \in [0, 2s), \gamma_0 \in [0, \gamma]$ and $0 < c < C$, where $B_r = B_r(0)$. Furthermore, for any given solution to the differential equation in (1.1), namely without a prescribed initial datum, we also prove that there exists a unique initial trace which is a positive finite Radon measure (see Theorem 3.3). Observe that this result suggests that is quite natural to consider a positive finite Radon measure μ as the initial condition in (1.1). We stress that the results concerning uniqueness are new even for $\rho \equiv 1$, which obviously fulfills (1.2), thus solving an open problem posed in [42] where such a problem is addressed for initial data given by Dirac deltas, namely for *Barenblatt solutions*. In this case, the problem is known as *fractional porous media equation* and has been thoroughly analysed in [17, 18] for initial data in $L^1(\mathbb{R}^d)$. More in general, in view of various applications well outlined in the literature (see e.g. [26]), we also consider the weight $\rho(x)$ since the same methods of proof work in this case as well. In this regard, observe that even if $\rho \in C(\mathbb{R}^d)$ has a suitable decay at infinity, and $\mu = u_0 \in L^1_\rho(\mathbb{R}^d)$, then the asymptotics of *any* solution can be determined by referring to the *Barenblatt solution* (i.e. the solution to problem (1.1) with $\mu = \delta$) for the problem with *singular, homogeneous* weight $\rho(x) = |x|^{-\gamma}$, which makes the latter scale-invariant. Also for this reason we treat weights ρ that satisfy (1.2), thus being allowed to be singular at $x = 0$. However, some further restrictions on s, d and γ will be required and clarified later, see Theorems 3.2 and 3.4. Let us mention that our results entailing the existence and uniqueness of Barenblatt solutions for singular weights are used in a crucial way in [24] to obtain the asymptotic behavior recalled above.

The analysis of the evolutions addressed here poses significant difficulties especially as concerns uniqueness, as can be guessed even when considering their linear analogues. In fact, the first issue

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we have to deal with is the essential self-adjointness of the operator formally defined as $\rho^{-1}(-\Delta)^s$ on test functions, and the validity of the Markov property for the associated linear evolution. This will be crucial in the uniqueness proof and holds only if γ is not too large. For larger γ one expects that suitable conditions at infinity should be required to recover self-adjointness.

Notice that the study of weighted linear differential operators of second order has a long story, see for example [13, Section 4.7] or [32]. Recently, the analysis of the spectral properties of operators which are modeled on the critical operator formally given by $|x|^2\Delta$ has been performed in [14].

As for nonlinear evolutions, the study of porous media and fast diffusion equations with measure data can be tracked back to the pioneering papers [2, 7, 34, 11]. See [43, Section 13] for details and additional references. The fast diffusion case, which will not be dealt with here, is investigated in [8, 9]: notice that for such evolutions the Dirac delta may not be smoothed into a regular solution, so that different techniques must be used, see the recent paper [35] for a general approach. In [17, 18], the fractional porous media and fast diffusion equations have been introduced and thoroughly studied for initial data which are integrable functions. The construction of Barenblatt solutions and the analysis of their role as asymptotic attractors for general integrable data is performed in [42]. Existence and uniqueness of solutions in the fractional, weighted case is studied in [37, 38]: however, the weight there cannot be singular and data cannot be measures.

Semilinear heat equations with measure data have a long history as well and have recently been studied also in the fractional case, see e.g. [29, 10] and references quoted. We remark that the terminology "measure data" is sometimes used in different contexts in which a measure appears as a source term in certain evolution equations: see e.g. [30].

There is a huge literature on the weighted porous media equation: see for example [15, 16, 20, 21, 22, 23, 25, 26, 27, 36, 39, 40, 41] and references quoted therein. It should be pointed out that the possible singularity of the weight, and the fact that we consider measure data as well, makes our problem significantly different both from the non-weighted, fractional case and from the weighted, non-fractional case: straightforward modifications of the strategies used to tackle such problems turn out not to be applicable here.

Finally, notice that fractional porous media equations are being used as a model in several applied contexts, see e.g. [5, Appendix B] and references quoted for details.

Outline of the paper. The paper is organized as follows. Section 2 briefly collects some preliminary tools on measure theory, fractional Laplacians and fractional Sobolev spaces. In Section 3 we state our main results. In Section 4 we prove existence of weak solutions and the result concerning existence and uniqueness of the initial trace, whereas in Section 5 uniqueness, which is by far the most delicate issue, is addressed: notice that, although we do not state this explicitly, the proofs work also in the case $s = 1$ and the corresponding results are new in this context as well for the weighted case. In proving uniqueness, we use a "duality method", following the same line of reasoning introduced by M. Pierre in [34]. This entails serious new difficulties due to the presence of the fractional diffusion and of the weight ρ. In Appendix A we recall some technical results on the fractional Laplacian, which are exploited in several approximating procedures developed in the proofs below. In Appendix B we sketch the proof of the main properties of the linear operator formally given by $\rho^{-1}(-\Delta)^s$. Such properties are of independent interest but are also crucial in order to establish uniqueness.

2. Preliminary tools

In this section we outline some basic notation, definitions and properties that we shall make us of later, which concern weighted Lebesgue spaces, measures, fractional Laplacians, fractional Sobolev spaces and Riesz potentials of measures.

Weighted Lebesgue spaces. For a given measurable function $\rho : \mathbb{R}^d \to \mathbb{R}^+$ (that is, a weight), we denote as $L^p_\rho(\mathbb{R}^d)$ (let $p \in [1, \infty)$) the Banach space constituted by all (classes of equivalence of) measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$\|f\|_{p,\rho} := \left(\int_{\mathbb{R}^d} |f(x)|^p \, \rho(x) \mathrm{d}x \right)^{1/p} < \infty \,.$$

In the special case $\rho(x) = |x|^\alpha$ (let $\alpha \in \mathbb{R}$) we simplify notation and replace $L^p_\rho(\mathbb{R}^d)$ by $L^p_\alpha(\mathbb{R}^d)$ and $\|f\|_{p,\rho}$ by $\|f\|_{p,\alpha}$. For the usual unweighted Lebesgue spaces we keep the symbol $L^p(\mathbb{R}^d)$, denoting the corresponding norms as $\|f\|_p$ or $\|f\|_{L^p(\mathbb{R}^d)}$.

Positive finite Radon measures on $\mathbb{R}^d$. Since in (1.1) we deal with positive finite Radon measures μ on $\mathbb{R}^d$, we recall some basic properties enjoyed by the set of such measures, which we denote as $\mathcal{M}(\mathbb{R}^d)$ (with a slight abuse of notation: this is the usual symbol for the space of *signed* measures on $\mathbb{R}^d$). To begin with, consider a sequence $\{\mu_n\} \subset \mathcal{M}(\mathbb{R}^d)$. Following the notation of [34], we say that $\{\mu_n\}$ converges to $\mu \in \mathcal{M}(\mathbb{R}^d)$ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ if there holds

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \phi \, \mathrm{d}\mu_n = \int_{\mathbb{R}^d} \phi \, \mathrm{d}\mu \quad \forall \phi \in C_c(\mathbb{R}^d) \,, \tag{2.1}$$

where $C_c(\mathbb{R}^d)$ is the space of continuous, compactly supported functions on $\mathbb{R}^d$. This is usually referred to as *local weak* convergence* (see [1, Definition 1.58]). A classical theorem in measure theory asserts that if

$$\sup_n \mu_n(\mathbb{R}^d) < \infty \tag{2.2}$$

then there exists $\mu \in \mathcal{M}(\mathbb{R}^d)$ such that $\{\mu_n\}$ converges to μ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ up to subsequences (see [1, Theorem 1.59]). The same holds if we replace $C_c(\mathbb{R}^d)$ with $C_0(\mathbb{R}^d)$, the latter being the closure of the former w.r.t. $\|\cdot\|_\infty$. A stronger notion of convergence is the following. A sequence $\{\mu_n\} \subset \mathcal{M}(\mathbb{R}^d)$ is said to converge to $\mu \in \mathcal{M}(\mathbb{R}^d)$ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ if

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \phi \, \mathrm{d}\mu_n = \int_{\mathbb{R}^d} \phi \, \mathrm{d}\mu \quad \forall \phi \in C_b(\mathbb{R}^d) \,, \tag{2.3}$$

where $C_b(\mathbb{R}^d)$ is the space of continuous, bounded functions on $\mathbb{R}^d$. Trivially, (2.3) implies (2.1). The opposite holds under a further hypothesis. That is, if $\{\mu_n\}$ converges to μ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ and $\lim_{n\to\infty} \mu_n(\mathbb{R}^d) = \mu(\mathbb{R}^d)$, then $\{\mu_n\}$ converges to μ also in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ (see [1, Proposition 1.80]). Notice that if $\{\mu_n\}$ converges to μ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ and (2.2) holds, a priori one only has a weak* lower semi-continuity property:

$$\mu(\mathbb{R}^d) \le \liminf_{n\to\infty} \mu_n(\mathbb{R}^d)$$

(see again [1, Theorem. 1.59]).

Fractional Laplacians and fractional Sobolev spaces. The fractional s-Laplacian operator which appears in (1.1) is defined, at least for any $\phi \in \mathcal{D}(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d)$, as

$$(-\Delta)^s(\phi)(x) := p.v. \ C_{d,s} \int_{\mathbb{R}^d} \frac{\phi(x) - \phi(y)}{|x-y|^{d+2s}} \, \mathrm{d}y \quad \forall x \in \mathbb{R}^d \,,$$

where $C_{d,s}$ is a suitable positive constant depending only on d and s. However, since a priori we have no clue about the regularity of solutions to (1.1), it is necessary to reformulate the problem in a suitable weak sense, see Definition 3.1 below. Before doing it, we need to introduce some fractional Sobolev spaces. Here we shall mainly deal with $\dot{H}^s(\mathbb{R}^d)$, that is the closure of $\mathcal{D}(\mathbb{R}^d)$ w.r.t. the norm

$$\|\phi\|^2_{\dot{H}^s} := \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\phi(x) - \phi(y))^2}{|x-y|^{d+2s}} \, \mathrm{d}x \mathrm{d}y \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d) \,.$$

Notice that the space usually denoted as $H^s(\mathbb{R}^d)$ is just $L^2(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$. For definitions and properties of the general fractional Sobolev spaces $W^{r,p}(\mathbb{R}^d)$ we refer the reader e.g. to [19].

The link between the s-Laplacian and the space $\dot{H}^s(\mathbb{R}^d)$ can be seen by means of the identity

$$\begin{aligned}\frac{C_{d,s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(\phi(x)-\phi(y))(\psi(x)-\psi(y))}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^d}(-\Delta)^{\frac{s}{2}}(\phi)(x)\,(-\Delta)^{\frac{s}{2}}(\psi)(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d}\phi(x)(-\Delta)^s(\psi)(x)\,\mathrm{d}x\end{aligned} \tag{2.4}$$

for all $\phi,\psi\in\mathcal{D}(\mathbb{R}^d)$, see [19, Section 3]. In particular, $\|\phi\|_{\dot{H}^s}^2=\left\|(-\Delta)^{\frac{s}{2}}(\phi)\right\|_{L^2}^2$ for all $\phi\in\mathcal{D}(\mathbb{R}^d)$. Notice that (2.4) can be shown to hold, by approximation, also when $\phi\in\mathcal{D}(\mathbb{R}^d)$ is replaced by any $v\in\dot{H}^s(\mathbb{R}^d)$, where $(-\Delta)^{\frac{s}{2}}(v)$ is meant in the sense of distributions. By a further approximation procedure one then gets

$$\frac{C_{d,s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y=\int_{\mathbb{R}^d}(-\Delta)^{\frac{s}{2}}(v)(x)\,(-\Delta)^{\frac{s}{2}}(w)(x)\,\mathrm{d}x \quad \forall v,w\in\dot{H}^s(\mathbb{R}^d)\,. \tag{2.5}$$

If we set $v=w$ in (2.5) we deduce that $\|v\|_{\dot{H}^s}^2=\left\|(-\Delta)^{\frac{s}{2}}(v)\right\|_{L^2}^2$ also for any $v\in\dot{H}^s(\mathbb{R}^d)$. In Sections 4 and 5 (and in Appendix B) we shall deal with functions which belong to $\dot{H}^s(\mathbb{R}^d)$ and to weighted Lebesgue spaces.

Riesz potentials. Another mathematical object deeply linked with the s-Laplacian is its Riesz kernel, namely the function

$$I_{2s}(x):=\frac{k_{d,s}}{|x|^{d-2s}}\,,$$

where $k_{d,s}$ is again a positive constant depending only on d and s. For a given (possibly signed) finite Radon measure ν, one can show that the convolution

$$U^\nu:=I_{2s}*\nu$$

yields an $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ function referred to as the *Riesz potential* of ν, which formally satisfies

$$(-\Delta)^s(U^\nu)=\nu\,.$$

That is, still at a formal level, the convolution against I_{2s} coincides with the operator $(-\Delta)^{-s}$. One of the most important and classical references for Riesz potentials is the monograph [28] by N. S. Landkof. In the proof of Theorem 3.2 and throughout Section 5 we shall exploit some crucial properties of Riesz potentials collected in [28], along with their connections with the s-Laplacian.

3. Statements of the main results

We start by introducing a suitable notion of weak solution to (1.1), in the spirit of [18] and [38].

Definition 3.1. *Given a finite positive finite Radon measure μ, by a weak solution to problem* (1.1) *we mean a nonnegative function u such that*

$$u\in L^\infty((0,\infty);L^1_\rho(\mathbb{R}^d))\cap L^\infty(\mathbb{R}^d\times(\tau,\infty)) \quad \forall\tau>0\,, \tag{3.1}$$

$$u\in L^2_{\mathrm{loc}}((0,\infty);\dot{H}^s(\mathbb{R}^d))\,, \tag{3.2}$$

$$-\int_0^\infty\int_{\mathbb{R}^d}u(x,t)\varphi_t(x,t)\,\rho(x)\mathrm{d}x\mathrm{d}t+\int_0^\infty\int_{\mathbb{R}^d}(-\Delta)^{\frac{s}{2}}(u^m)(x,t)\,(-\Delta)^{\frac{s}{2}}(\varphi)(x,t)\,\mathrm{d}x\mathrm{d}t=0 \quad \forall\varphi\in C_c^\infty(\mathbb{R}^d\times(0,\infty)) \tag{3.3}$$

and

$$\operatorname*{ess\,lim}_{t\to0}\rho(\cdot)\,u(\cdot,t)=\mu \quad \text{in } \sigma(\mathcal{M}(\mathbb{R}^d),C_b(\mathbb{R}^d))\,. \tag{3.4}$$

Our first result concerns existence.

Theorem 3.2. *Let $d > 2s$ and assume that ρ satisfies* (1.2) *for some $\gamma \in [0, 2s) \cap [0, d-2s]$ and $\gamma_0 \in [0, \gamma]$. Let μ be a positive finite Radon measure. Then there exists a weak solution u to* (1.1) *according to Definition 3.1, which conserves the mass in the sense that $\mu\left(\mathbb{R}^d\right) = \int_{\mathbb{R}^d} u(x,t)\rho(x)\mathrm{d}x$ for all $t > 0$, and satisfies the smoothing effect*

$$\|u(t)\|_\infty \leq K\, t^{-\alpha}\, \mu(\mathbb{R}^d)^\beta \quad \forall t > 0\,, \tag{3.5}$$

where K depends only on m, γ, s, d and on the constant C appearing in (1.2), *and*

$$\alpha := \frac{d-\gamma}{(m-1)(d-\gamma)+2s-\gamma}\,, \quad \beta := \frac{2s-\gamma}{(m-1)(d-\gamma)+2s-\gamma}\,.$$

In particular, $u(\cdot, t) \in L^p_\rho(\mathbb{R}^d)$ for all $t > 0$ and $p \in [1, \infty]$. In addition, the solution satisfies the energy estimates

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \left|(-\Delta)^{\frac{s}{2}}\left(u^m\right)(x,t)\right|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{m+1}\int_{\mathbb{R}^d} u^{m+1}(x,t_2)\,\rho(x)\mathrm{d}x = \frac{1}{m+1}\int_{\mathbb{R}^d} u^{m+1}(x,t_1)\,\rho(x)\mathrm{d}x \tag{3.6}$$

and

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^d} |z_t(x,t)|^2\ \rho(x)\mathrm{d}x\mathrm{d}t \leq C' \int_{\mathbb{R}^d} u^{m+1}\left(x, t_1/2\right)\ \rho(x)\mathrm{d}x \tag{3.7}$$

for all $t_2 > t_1 > 0$, where $z := u^{\frac{m+1}{2}}$ and C' depends on m, t_1 and t_2.

The method of proof of Theorem 3.2 allows us to prove the following result on existence and uniqueness of the initial trace, in the spirit of [5, Section 7] and [3].

Theorem 3.3. *Let $d > 2s$ and assume that ρ satisfies* (1.2) *for some $\gamma \in [0, 2s) \cap [0, d-2s]$ and $\gamma_0 \in [0, \gamma]$. Consider a weak solution u to $\rho(x)u_t + (-\Delta)^s\left(u^m\right) = 0$ in the sense that u satisfies* (3.1), (3.2) *and* (3.3). *Then there exists a unique positive finite Radon measure μ which is the initial trace of u in the sense of* (3.4). *The same result holds if the condition $u \in L^\infty(\mathbb{R}^d \times (\tau, \infty))$ in* (3.1) *is replaced by the weaker condition $\int_{t_1}^{t_2} u^m(\cdot, \tau)\,\mathrm{d}\tau \in L^1_\rho(\mathbb{R}^d)$ for all $t_2 > t_1 > 0$. In particular, $\mu\left(\mathbb{R}^d\right) = \int_{\mathbb{R}^d} u(x,t)\rho(x)\mathrm{d}x$ for all $t > 0$.*

As for uniqueness of weak solutions we have the next result.

Theorem 3.4. *Let $d > 2s$ and assume that ρ satisfies* (1.2) *for some $\gamma \in [0, 2s) \cap [0, d-2s]$ and $\gamma_0 \in [0, \gamma]$. Let u_1, u_2 be two weak solutions to* (1.1) *in the sense of Definition 3.1. Suppose that they take as initial datum the same positive finite Radon measure μ, in the sense of* (3.4). *Then $u_1 = u_2$.*

Remark 3.5. Notice that, if $d \geq 4s$, then the assumptions on γ reduce to $\gamma \in [0, 2s)$.

Let us stress that, in order to prove Theorem 3.4, we shall crucially exploit the properties of the operator $A = \rho^{-1}\,(-\Delta)^s$ contained in Theorem 3.7 and Proposition B.1 below. Such results are of independent interest; their proofs will be just sketched, to keep the paper in a reasonable length, in Appendix B. Some further details and extentions are given in [31].

Definition 3.6. *Let $d > 2s$ and assume that ρ satisfies* (1.2) *for some $\gamma \in [0, 2s)$ and $\gamma_0 \in [0, d)$. We denote as $X_{s,\rho}$ the Hilbert space of all functions $v \in L^2_\rho(\mathbb{R}^d)$ such that $(-\Delta)^s(v)$ (as a distribution) belongs to $L^2_{\rho^{-1}}(\mathbb{R}^d)$, equipped with the norm*

$$\|v\|^2_{X_{s,\rho}} := \|v\|^2_{2,\rho} + \|(-\Delta)^s(v)\|^2_{2,\rho^{-1}} \quad \forall v \in X_{s,\rho}\,.$$

Theorem 3.7. *Let $d > 2s$ and assume that ρ satisfies* (1.2) *for some $\gamma \in [0, 2s)$ and $\gamma_0 \in [0, d)$. Let $A : D(A) := X_{s,\rho} \subset L^2_\rho(\mathbb{R}^d) \to L^2_\rho(\mathbb{R}^d)$ be the operator*

$$A(v) := \rho^{-1}\,(-\Delta)^s(v) \quad \forall v \in X_{s,\rho}\,.$$

Then A is densely defined, positive and self-adjoint on $L^2_\rho(\mathbb{R}^d)$, and the quadratic form associated to it is

$$Q(v,v) := \frac{C_{d,s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(v(x)-v(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y$$

with domain $D(Q) := L^2_\rho(\mathbb{R}^d)\cap \dot{H}^s(\mathbb{R}^d)$. Moreover, Q is a Dirichlet form on $L^2_\rho(\mathbb{R}^d)$ and A generates a Markov semigroup $S_2(t)$ on $L^2_\rho(\mathbb{R}^d)$. In particular, for all $p \in [1,\infty]$ there exists a contraction semigroup $S_p(t)$ on $L^p_\rho(\mathbb{R}^d)$, consistent with $S_2(t)$ on $L^2_\rho(\mathbb{R}^d)\cap L^p_\rho(\mathbb{R}^d)$, which is furthermore analytic with a suitable angle $\theta_p > 0$ for $p \in (1,\infty)$.

4. Existence of weak solutions

We start showing a direct consequence of Definition 3.1, namely the conservation in time of the "mass" $\int_{\mathbb{R}^d} u(x,t)\,\rho(x)\mathrm{d}x$ (recall that we are considering nonnegative solutions).

Proposition 4.1. Let $d > 2s$ and assume that ρ satisfies (1.2) for some $\gamma \in [0,2s)\cap[0,d-2s]$ and $\gamma_0 \in [0,\gamma]$. Let u be a weak solution to (1.1) according to Definition 3.1. Then

$$\|u(t)\|_{1,\rho} = \int_{\mathbb{R}^d} u(x,t)\,\rho(x)\mathrm{d}x = \mu(\mathbb{R}^d) \quad \text{for a.e. } t>0\,, \tag{4.1}$$

namely we have *conservation of mass.*

Proof. We plug into (3.3) the test function $\varphi_R(x,t) := \vartheta(t)\xi_R(x)$, where ξ_R is the same cut-off function as in Lemma A.3 and ϑ is a suitable positive, regular and compactly supported approximation of $\chi_{[t_1,t_2]}$ (let $t_2 > t_1 > 0$). Using (2.5), Lemma A.1, Lemma A.3 and letting $\vartheta \to \chi_{[t_1,t_2]}$ in (3.3), it is straightforward to obtain the following estimate:

$$\begin{aligned}&\left|\int_{\mathbb{R}^d} u(x,t_2)\xi_R(x)\,\rho(x)\mathrm{d}x - \int_{\mathbb{R}^d} u(x,t_1)\xi_R(x)\,\rho(x)\mathrm{d}x\right|\\ \le\, & c^{-1}\left(\frac{1}{R^{2s}}+\frac{1}{R^{2s-\gamma}}\right)\|(1+|x|^\gamma)(-\Delta)^s(\xi)\|_\infty \int_{t_1}^{t_2}\int_{\mathbb{R}^d} u^m(x,t)\,\rho(x)\mathrm{d}x\mathrm{d}t\,,\end{aligned} \tag{4.2}$$

where on the r.h.s. we exploited the inequality $\rho^{-1}(x) \le c^{-1}(1+|x|^\gamma)$ for all $x \in \mathbb{R}^d$, direct consequence of (1.2). Letting $R \to \infty$ in (4.2) and recalling (3.4) we get the conclusion. □

The proof of existence of weak solutions to (1.1) is based on an approximation procedure, that is on picking a sequence of initial data in $L^1_\rho(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ which suitably converges to μ. An additional approximation will be needed to deal with the possible singularity of the weight at the origin. The corresponding approximate problems are addressed in the next subsection. Since the procedure is in principle standard although technically delicate, we underline the main points only.

4.1. Approximate problems with initial data in $L^1_\rho(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$.

We are concerned with existence of solutions to the following problem:

$$\begin{cases}\rho(x)u_t + (-\Delta)^s(u^m) = 0 & \text{in } \mathbb{R}^d\times\mathbb{R}^+\,,\\ u = u_0 & \text{on } \mathbb{R}^d\times\{0\}\,.\end{cases} \tag{4.3}$$

Such solutions are meant in the sense of Definition 3.1 with μ replaced by ρu_0.

Lemma 4.2. *Let $d > 2s$ and assume that ρ satisfies* (1.2) *for some $\gamma \in [0,2s)\cap[0,d-2s]$ and $\gamma_0 \in [0,\gamma]$. Let $u_0 \in L^1_\rho(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, with $u_0 \ge 0$. Then there exists a weak solution u to* (4.3) *which satisfies the energy estimates* (3.6), (3.7) *with a constant C' depending only on m, t_1 and t_2.*

Let us outline the strategy of the proof. We further approximate the problem (4.3) by regularizing the weight $\rho(x)$ in a neighbourhood of $x=0$ (where it can be singular). More precisely, we introduce for any $\eta > 0$ the following problem:

$$\begin{cases}\rho_\eta(x)\left(u_\eta\right)_t + (-\Delta)^s\left(u_\eta^m\right) = 0 & \text{in } \mathbb{R}^d\times\mathbb{R}^+\,,\\ u_\eta = u_0 & \text{on } \mathbb{R}^d\times\{0\}\,,\end{cases} \tag{4.4}$$

where $\{\rho_\eta\} \subset C(\mathbb{R}^d)$ is a family of strictly positive weights which behave like $|x|^{-\gamma}$ at infinity and approximate $\rho(x)$ monotonically from below, as $\eta \to 0$. Existence (and uniqueness) of weak solutions to (4.4) for such weights and initial data have been established in [38, Theorem 3.1]. We get suitable a priori estimates (namely (3.6) applied to u_η, which will be proved later, and (4.7) below), that enable us to pass to the limit as $\eta \to 0$, and obtain a solution to problem (4.3), by standard compactness arguments.

Proof. For any $\eta > 0$ let u_η be the unique solution to problem (4.4). Such solutions belong to $C([0,\infty); L^1_{\rho_\eta}(\mathbb{R}^d))$ and satisfy the bound $\|u_\eta\|_{L^\infty(\mathbb{R}^d\times(0,\infty))} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}$. Exploiting these properties one can show that each u_η satisfies a weak formulation which is slightly stronger than the one of Definition 3.1:

$$\begin{aligned} &-\int_0^T\int_{\mathbb{R}^d} u_\eta(x,t)\varphi_t(x,t)\,\rho_\eta(x)\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(u_\eta^m)(x,t)\,(-\Delta)^{\frac{s}{2}}(\varphi)(x,t)\,\mathrm{d}x\mathrm{d}t \\ &= \int_{\mathbb{R}^d} u_0(x)\varphi(x,0)\,\rho_\eta(x)\mathrm{d}x \end{aligned} \tag{4.5}$$

for all $T > 0$ and $\varphi \in C_c^\infty(\mathbb{R}^d \times [0,T))$ (so that $\varphi(\cdot,T) = 0$), where $u_\eta^m \in L^2((0,\infty); \dot{H}^s(\mathbb{R}^d))$. The latter property follows from the validity of the energy identity (3.6) for u_η for all $t_2 > t_1 \ge 0$. Formally, (3.6) can be proved by plugging the test function $\varphi(x,t) := \vartheta(t)u_\eta^m(x,t)$ into the weak formulation (4.5) and letting ϑ tend to $\chi_{[t_1,t_2]}$ as in the proof of Proposition 4.1. In order to justify rigorously the validity of (3.6) for u_η, one must proceed as in [18, Section 8]. A crucial point concerns the fact that our solutions are *strong*, which follows by techniques analogous to the ones used in [18, Section 8.1]. We refer the reader to Section 4.5 below for more details. We have:

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^d} |(z_\eta)_t(x,t)|^2\,\rho_\eta(x)\mathrm{d}x\mathrm{d}t \le C\int_{\mathbb{R}^d} u_\eta^{m+1}(x,t_1/2)\,\rho(x)\mathrm{d}x \quad \forall t_2 > t_1 > 0\,, \tag{4.6}$$

where $z_\eta := u_\eta^{\frac{m+1}{2}}$ and C depends only on m, t_1 and t_2. Formula (4.6) follows as in [18, Lemma 8.1]. Since

$$\left(u_\eta^m\right)_t = c_m\, z_\eta^{\frac{m-1}{m+1}}\,(z_\eta)_t \quad \text{and} \quad \|z_\eta\|_{L^\infty(\mathbb{R}^d\times(0,\infty))} = \|u_\eta\|_{L^\infty(\mathbb{R}^d\times(0,\infty))}^{\frac{m+1}{2}} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}^{\frac{m+1}{2}}\,,$$

from (4.6) we deduce that

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \left|\left(u_\eta^m\right)_t(x,t)\right|^2 \rho_\eta(x)\mathrm{d}x\mathrm{d}t \le k\,\|u_0\|_\infty^{m-1} \quad \forall t_2 > t_1 > 0 \tag{4.7}$$

for a suitable $k > 0$ independent of η. Moreover, the validity of $\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\left|u_\eta^m(x,t)\right|^2\rho_\eta(x)\mathrm{d}x\mathrm{d}t \le C''$ for all $t_2 > t_1 \ge 0$ and for another suitable positive constant C'' that depends only on m, t_1, t_2 and u_0 is ensured by the conservation of mass (4.1) (with $\rho = \rho_\eta$) and by the uniform bound on $\|u_\eta\|_{L^\infty(\mathbb{R}^d\times(0,\infty))}$. Let $n \in \mathbb{N}$. We now use (A.4) with $\xi_1 = \xi_{1,n} \in C^\infty(\mathbb{R}^d)$ such that

$$\xi_1 \equiv 1 \quad \text{in } B_n\,, \quad \xi_1 \equiv 0 \quad \text{in } B_{2n}\,,$$

and with $\xi_2 = \xi_{2,n} \in C^\infty((0,\infty))$ such that

$$\xi_2 \equiv 1 \quad \text{in } \left(\frac{1}{n}, n\right), \quad \xi_2 \equiv 0 \quad \text{in } \left(0, \frac{1}{2n}\right) \cup (2n,\infty)\,.$$

The fact that $H^s(\mathbb{R}^{d+1})$ is compactly embedded in $L^2_{\mathrm{loc}}(\mathbb{R}^{d+1})$ (see e.g. [19, Theorem 7.1]), and a standard diagonal procedure allow us to pass to the limit as $\eta \to 0$ in (4.5) and get that the weak limit u of $\{u_\eta\}$ satisfies

$$\begin{aligned} &-\int_0^T\int_{\mathbb{R}^d} u(x,t)\varphi_t(x,t)\,\rho(x)\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(u^m)(x,t)\,(-\Delta)^{\frac{s}{2}}(\varphi)(x,t)\,\mathrm{d}x\mathrm{d}t \\ &= \int_{\mathbb{R}^d} u_0(x)\varphi(x,0)\,\rho(x)\mathrm{d}x \end{aligned} \tag{4.8}$$

for all $T > 0$ and $\varphi \in C_c^\infty(\mathbb{R}^d \times [0,T))$. The validity of (3.4) follows by plugging into (4.8) the test function $\varphi(x,t) := \vartheta(t)\xi_R(x)$, where ξ_R is a cut-off function as in Lemma A.3 and ϑ is a regular approximation of $\chi_{[0,t_2]}$. One then lets $t_2 \to 0$ and $R \to \infty$.

The energy estimates (3.6) and (3.7) for u can be obtained reasoning exactly as above (one uses again the fact that solutions are strong). □

4.2. **Stroock-Varopoulos inequality and smoothing estimate.** Having at our disposal an existence result for problem (4.3), we can now let ρu_0 approximate μ. In order to show that the corresponding solutions converge to a solution of (1.1), we need first some technical results. We begin with a modification of the classical Stroock-Varopoulos inequality: it is proved here for $v \in L^\infty(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$ with $(-\Delta)^s(v) \in L^1(\mathbb{R}^d)$. Observe that, under the hypothesis that $v \in L^q(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$ with $(-\Delta)^s(v) \in L^q(\mathbb{R}^d)$, for $q > 1$, such an inequality can be found, e.g., in [18, Section 5] or [4]. See also [13, formula (2.2.7)] for a similar inequality involving general Dirichlet forms. The present result seems to be new, in view of its functional framework, therefore its proof is given in some detail.

Lemma 4.3. *Let $d > 2s$. For all nonnegative $v \in L^\infty(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$ such that $(-\Delta)^s(v) \in L^1(\mathbb{R}^d)$, the inequality*

$$\int_{\mathbb{R}^d} v^{q-1}(x)(-\Delta)^s(v)(x)\,\mathrm{d}x \geq \frac{4(q-1)}{q^2} \int_{\mathbb{R}^d} \left|(-\Delta)^{\frac{s}{2}}\left(v^{\frac{q}{2}}\right)(x)\right|^2 \mathrm{d}x \tag{4.9}$$

holds for any $q > 1$.

Proof. We shall assume, with no loss of generality, that v is a regular function. Indeed, by standard mollification arguments, one can always pick a sequence $\{v_n\} \subset C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$ such that $\{v_n\}$ converges pointwise to v, $\|v_n\|_\infty \leq \|v\|_\infty$ and $\{(-\Delta)^s(v_n)\}$ converges to $(-\Delta)^s(v)$ in $L^1(\mathbb{R}^d)$. This is enough to pass to the limit as $n \to \infty$ on the l.h.s. of (4.9), while on the r.h.s. one exploits the weak lower semi-continuity of the L^2 norm.

Consider the following sequences of functions:

$$\psi_n(x) := \int_0^{x \wedge \frac{1}{n}} y^{\frac{4s}{d-2s}}\,\mathrm{d}y + (q-1)\int_{\frac{1}{n}}^{x \vee \frac{1}{n}} y^{q-2}\,\mathrm{d}y \quad \forall x \in \mathbb{R}^+ ,$$
$$\Psi_n(x) := \int_0^{x \wedge \frac{1}{n}} y^{\frac{2s}{d-2s}}\,\mathrm{d}y + (q-1)^{\frac{1}{2}}\int_{\frac{1}{n}}^{x \vee \frac{1}{n}} y^{\frac{q}{2}-1}\,\mathrm{d}y \quad \forall x \in \mathbb{R}^+ .$$

It is plain that ψ_n and Ψ_n are absolutely continuous, monotone increasing functions such that $\psi_n'(x) = [\Psi_n'(x)]^2$ for all $x \in \mathbb{R}^+$. For any $R > 0$, take a cut-off function ξ_R as in Lemma A.3. To the function $\xi_R v$ one can apply Lemma 5.2 of [18] with the choices $\psi = \psi_n$ and $\Psi = \Psi_n$, which yields

$$\int_{\mathbb{R}^d} \psi_n(\xi_R v)(x)\,(-\Delta)^s(\xi_R v)(x)\,\mathrm{d}x \geq \int_{\mathbb{R}^d} \left|(-\Delta)^{\frac{s}{2}}(\Psi_n(\xi_R v))(x)\right|^2 \mathrm{d}x . \tag{4.10}$$

Expanding the s-Laplacian of the product of two functions, we get that the l.h.s. of (4.10) equals

$$\begin{aligned} &\int_{\mathbb{R}^d} \psi_n(\xi_R v)(x)\,\xi_R(x)(-\Delta)^s(v)(x)\,\mathrm{d}x + \int_{\mathbb{R}^d} \psi_n(\xi_R v)(x)(-\Delta)^s(\xi_R)(x)v(x)\,\mathrm{d}x \\ &+ 2\,C_{d,s}\int_{\mathbb{R}^d} \psi_n(\xi_R v)(x)\int_{\mathbb{R}^d} \frac{(\xi_R(x)-\xi_R(y))(v(x)-v(y))}{|x-y|^{d+2s}}\,\mathrm{d}y\mathrm{d}x . \end{aligned} \tag{4.11}$$

By dominated convergence,

$$\lim_{R\to\infty} \int_{\mathbb{R}^d} \psi_n(\xi_R v)(x)\,\xi_R(x)(-\Delta)^s(v)(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \psi_n(v)(x)(-\Delta)^s(v)(x)\,\mathrm{d}x .$$

Our aim is to show that the other two integrals in (4.11) go to zero as $R \to \infty$. We have:

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} \psi_n(\xi_R v)(x)(-\Delta)^s(\xi_R)(x) v(x)\,\mathrm{d}x \right| \\ &\le \|(-\Delta)^s(\xi_R)\|_\infty \left(\frac{d-2s}{d+2s} \int_{\{v \le \frac{1}{n}\}} v^{\frac{2d}{d-2s}}(x)\,\mathrm{d}x + \psi_n(\|v\|_\infty)\|v\|_\infty \int_{\{v > \frac{1}{n}\}} \mathrm{d}x \right) \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} \psi_n(\xi_R v)(x) \int_{\mathbb{R}^d} \frac{(\xi_R(x)-\xi_R(y))(v(x)-v(y))}{|x-y|^{d+2s}}\,\mathrm{d}y\mathrm{d}x \right| \\ &\le \|v\|_{\dot{H}^s} \left(\int_{\mathbb{R}^d} [\psi_n(\xi_R v)(x)]^2 \int_{\mathbb{R}^d} \frac{(\xi_R(x)-\xi_R(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}y\mathrm{d}x \right)^{\frac{1}{2}} \\ &\le \|v\|_{\dot{H}^s} \|l_s(\xi_R)\|_\infty^{\frac{1}{2}} \left(\left[\frac{d-2s}{d+2s} \right]^2 \int_{\{v \le \frac{1}{n}\}} v^{2\frac{d+2s}{d-2s}}(x)\,\mathrm{d}x + [\psi_n(\|v\|_\infty)]^2 \int_{\{v > \frac{1}{n}\}} \mathrm{d}x \right)^{\frac{1}{2}}, \end{aligned} \tag{4.13}$$

where l_s is defined in Lemma A.2. Thanks to the scaling properties of both $(-\Delta)^s(\xi_R)$ and $l_s(\xi_R)$ (Lemma A.3), it is immediate to check that $\lim_{R\to\infty} \|(-\Delta)^s(\xi_R)\|_\infty = \lim_{R\to\infty} \|l_s(\xi_R)\|_\infty = 0$. Moreover, notice that $v \in L^{\frac{2d}{d-2s}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ (see [19, Section 6] or Lemma 4.4 below). In particular, v also belongs to $L^{2\frac{d+2s}{d-2s}}(\mathbb{R}^d)$. Thus, letting $R \to \infty$ in (4.12) and (4.13), we deduce that the last two integrals in (4.11) vanish, so that we can pass to the limit on the l.h.s. of (4.10). On the r.h.s. we just use the fact that $(-\Delta)^{\frac{s}{2}}(\Psi_n(\xi_R v))$ converges to $(-\Delta)^{\frac{s}{2}}(\Psi_n(v))$ weakly in $L^2(\mathbb{R}^d)$. This proves the validity of

$$\int_{\mathbb{R}^d} \psi_n(v)(x)(-\Delta)^s(v)(x)\,\mathrm{d}x \ge \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{s}{2}}(\Psi_n(v))(x) \right|^2 \mathrm{d}x\,. \tag{4.14}$$

The final step is to let $n \to \infty$ in (4.14). It is clear that the sequence $\{\psi_n(x)\}$ converges locally uniformly to the function x^{q-1}, while $\{\Psi_n(x)\}$ converges locally uniformly to $2(q-1)^{\frac{1}{2}} x^{\frac{q}{2}}/q$. Hence, $\{\psi_n(v)\}$ and $\{\Psi_n(v)\}$ converge in $L^\infty(\mathbb{R}^d)$ to v^{q-1} and $2(q-1)^{\frac{1}{2}} v^{\frac{q}{2}}/q$, respectively. This is enough in order to pass to the limit in (4.14) and obtain (4.9). □

Lemma 4.4. *Let $d > 2s$ and assume that ρ satisfies* (1.2) *for some $\gamma \in [0, 2s) \cap [0, d-2s]$ and $\gamma_0 \in [0, \gamma]$. There exists a positive constant $C_{CKN} = C_{CKN}(C, \gamma, s, d)$ such that the Caffarelli-Kohn-Nirenberg-type inequalities*

$$\|v\|_{q,\rho} \le C_{CKN} \left\| (-\Delta)^{\frac{s}{2}}(v) \right\|_2^{\frac{1}{\alpha+1}} \|v\|_{p,\rho}^{\frac{\alpha}{\alpha+1}} \qquad \forall v \in L^p_\rho(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$$

hold for any $\alpha \ge 0$, $p \ge 1$ and $q = 2(d-\gamma)(\alpha+1)/\left[(d-\gamma)\frac{\alpha}{p} + d - 2s\right]$.

Proof. See e.g. [12, Theorem 1.8], where one considers the Sobolev inequality corresponding to $\alpha = 0$ here, and then uses an elementary interpolation. □

Lemmas 4.3 and 4.4 provide us with some functional inequalities which are crucial to prove the following *smoothing effect* for solutions to (4.3).

Proposition 4.5. *Let $d > 2s$ and assume that ρ satisfies* (1.2) *for some $\gamma \in [0, 2s) \cap [0, d-2s]$ and $\gamma_0 \in [0, \gamma]$. There exists a constant $K > 0$ depending only on m, γ, s, d and C such that, for all nonnegative initial datum $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and the corresponding weak solution u to* (4.3) *constructed in Lemma 4.2, the following $L^{p_0}_\rho$–L^∞ smoothing effect holds for any $p_0 \in [1, \infty)$:*

$$\|u(t)\|_\infty \le K\, t^{-\alpha_0} \, \|u_0\|_{p_0,\rho}^{\beta_0} \qquad \forall t > 0\,, \tag{4.15}$$

where

$$\alpha_0 := \frac{d-\gamma}{(m-1)(d-\gamma) + (2s-\gamma)p_0}\,, \qquad \beta_0 := \frac{(2s-\gamma)p_0}{(m-1)(d-\gamma) + (2s-\gamma)p_0}\,. \tag{4.16}$$

Proof. We omit the details, since the claim follows as in [18, Theorem 8.2] by means of a standard parabolic Moser iteration. Nevertheless, notice that the proof relies on the Stroock-Varopoulos inequality (which has to hold for the precise set of functions stated in Lemma 4.3), the Caffarelli-Kohn-Nirenberg type inequalities provided by Lemma 4.4 and the fact that the L^p_ρ norms do not increase along the evolution (see Section 4.5). □

4.3. **Proof of the existence result.** We outline the main steps of this proof. Suppose first that μ is compactly supported. Consider the family $\{u_\varepsilon\}$ of weak solutions to (1.1) that take on the regular initial data $\mu_\varepsilon := \psi_\varepsilon * \mu$ (let $\varepsilon > 0$), where $\psi_\varepsilon := \frac{1}{\varepsilon^d}\psi\left(\frac{x}{\varepsilon}\right)$ with $\psi \in \mathcal{D}_+(\mathbb{R}^d)$ and $\|\psi\|_1 = 1$. The existence of such family is ensured by Lemma 4.2, upon setting $u_0 = \rho^{-1}\mu_\varepsilon$. In view of certain a priori estimates (see (4.17), (4.18) and (4.19) below), we prove that $\{u_\varepsilon\}$ converges (up to subsequences), as $\varepsilon \to 0$, to a function u which satisfies (3.1), (3.2) and (3.3). Afterwards we deal with (3.4). To do this, we exploit some results in potential theory, following [34] or [42], using the Riesz potential $U_\varepsilon(\cdot,t)$ of $\rho(\cdot)u_\varepsilon(\cdot,t)$. Then we let $\varepsilon \to 0$; in doing this, a uniform estimate w.r.t. ε for the potentials (see (4.25) below) will be crucial. Finally, we consider general positive finite Radon measures μ, by a further approximation.

Proof of Theorem 3.2. For any $\varepsilon > 0$, let u_ε be as above. Combining the smoothing effect (4.15) with the fact that $\|\mu_\varepsilon\|_1 = \mu(\mathbb{R}^d)$ and with the conservation of mass (4.1), we obtain:

$$\int_{\mathbb{R}^d} u_\varepsilon^{m+1}(x,t)\,\rho(x)\mathrm{d}x \le \|u_\varepsilon(t)\|_\infty^m \,\|\mu_\varepsilon\|_1 \le K^m\, t^{-\alpha m}\,\mu(\mathbb{R}^d)^{1+\beta m} \tag{4.17}$$

for all $t > 0$. Hence, using (3.6), (3.7) and (4.17) we get:

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \left|(-\Delta)^{\frac{s}{2}}\left(u_\varepsilon^m\right)(x,t)\right|^2 \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^d} u_\varepsilon^{m+1}(x,t_2)\,\rho(x)\mathrm{d}x \le K^m\, t_1^{-\alpha m}\,\mu(\mathbb{R}^d)^{1+\beta m}\,, \tag{4.18}$$

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \left|(z_\varepsilon)_t\,(x,t)\right|^2 \rho(x)\mathrm{d}x\mathrm{d}t \le C' \int_{\mathbb{R}^d} u^{m+1}\left(x,t_1/2\right)\,\rho(x)\mathrm{d}x \tag{4.19}$$

for all $t_2 > t_1 > 0$, where $z_\varepsilon := u_\varepsilon^{\frac{m+1}{2}}$ and C' is a positive constant that depends on m, t_1, t_2 but is independent of ε. Thanks to (4.18), (4.19), the conservation of mass and the smoothing effect (which, in particular, bounds $\{u_\varepsilon\}$ in $L^\infty(\mathbb{R}^d \times (\tau,\infty))$ for all $\tau > 0$ independently of ε), we are allowed to proceed exactly as in the proof of Lemma 4.2. That is, we obtain that the pointwise limit u of $\{u_\varepsilon\}$, up to subsequences, satisfies (3.1), (3.2) and (3.3).

Let us now introduce the Riesz potential $U_\varepsilon(\cdot,t)$ of $\rho(\cdot)u_\varepsilon(\cdot,t)$. The equation solved by u_ε is

$$\rho(x)(u_\varepsilon)_t(x,t) = -\,(-\Delta)^s\,(u_\varepsilon^m)(x,t) \quad \forall (x,t) \in \mathbb{R}^d \times \mathbb{R}^+. \tag{4.20}$$

Applying to both sides of (4.20) the operator $(-\Delta)^{-s}$, namely the convolution against the Riesz kernel I_{2s} (recall the discussion in Section 2), formally yields

$$(U_\varepsilon)_t\,(x,t) = -u_\varepsilon^m(x,t) \quad \forall (x,t) \in \mathbb{R}^d \times \mathbb{R}^+\,. \tag{4.21}$$

To prove rigorously (4.21), we plug into (3.3) (with $u = u_\varepsilon$) the test function $\varphi(y,t) := \vartheta(t)\phi(y)$, where ϑ is a smooth and compactly supported approximation of $\chi_{[t_1,t_2]}$ and $\phi \in \mathcal{D}(\mathbb{R}^d)$. Integrating by parts (in space), letting ϑ tend to $\chi_{[t_1,t_2]}$ and replacing the function $\phi(y)$ by $\phi(y+x)$, with $x \in \mathbb{R}^d$ fixed, we get:

$$\begin{aligned}&\int_{\mathbb{R}^d} u_\varepsilon(y,t_2)\phi(y+x)\,\rho(y)\mathrm{d}y - \int_{\mathbb{R}^d} u_\varepsilon(y,t_1)\phi(y+x)\,\rho(y)\mathrm{d}y \\ &= -\int_{\mathbb{R}^d}\left(\int_{t_1}^{t_2} u_\varepsilon^m(y,t)\,\mathrm{d}t\right)(-\Delta)^s(\phi)(y+x)\,\mathrm{d}y\,.\end{aligned} \tag{4.22}$$

Integrating (4.22) against the Riesz kernel $I_{2s}(x)$ and using Fubini's Theorem gives (let $z = y + x$)

$$\begin{aligned}&\int_{\mathbb{R}^d} U_\varepsilon(z,t_2)\phi(z)\,\mathrm{d}z - \int_{\mathbb{R}^d} U_\varepsilon(z,t_1)\phi(z)\,\mathrm{d}z \\ &= -\int_{\mathbb{R}^d}\left(\int_{t_1}^{t_2} u_\varepsilon^m(y,t)\,\mathrm{d}t\right)\left(\int_{\mathbb{R}^d}(-\Delta)^s(\phi)(y+x)I_{2s}(x)\,\mathrm{d}x\right)\mathrm{d}y = -\int_{\mathbb{R}^d}\left(\int_{t_1}^{t_2} u_\varepsilon^m(y,t)\,\mathrm{d}t\right)\phi(y)\,\mathrm{d}y\,.\end{aligned} \tag{4.23}$$

The applicability of Fubini's Theorem is justified thanks to Lemma A.5, Lemma A.1 (recall that $d-2s \geq \gamma$ by assumption) and to the fact that $\int_{t_1}^{t_2} u_\varepsilon^m(\cdot,t)\,\mathrm{d}t$ belongs to $L^1_\rho(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ by (3.1).

By Lemma 4.2 and Definition 3.1, we know that $\rho u_\varepsilon(t)$ converges to μ_ε in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ as $t \to 0$. Hence, letting $t_1 \to 0$ in (4.23), we find that

$$\int_{\mathbb{R}^d} U_\varepsilon(x,t_2)\phi(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} U^{\mu_\varepsilon}(x)\phi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^d}\left(\int_0^{t_2} u_\varepsilon^m(x,t)\,\mathrm{d}t\right)\phi(x)\,\mathrm{d}x \tag{4.24}$$

for all $t_2 > 0$ and $\phi \in \mathcal{D}(\mathbb{R}^d)$. In fact,

$$\begin{aligned}\int_{\mathbb{R}^d} U_\varepsilon(x,t_1)\phi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} I_{2s}(x-y)\,\rho(y)u_\varepsilon(y,t_1)\,\mathrm{d}y\right)\phi(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d}\underbrace{\left(\int_{\mathbb{R}^d} I_{2s}(y-x)\,\phi(x)\,\mathrm{d}x\right)}_{U^\phi(y)}\rho(y)u_\varepsilon(y,t_1)\,\mathrm{d}y\,,\end{aligned}$$

and in view of Lemma A.5 we know that, in particular, $U^\phi \in C_0(\mathbb{R}^d)$, which allows to pass to the limit in the integral as $t_1 \to 0$. Thanks to the smoothing effect, the conservation of mass and the hypotheses on ρ, we can provide the following bound for (4.24):

$$\left|\int_{\mathbb{R}^d} U_\varepsilon(x,t_2)\phi(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} U^{\mu_\varepsilon}(x)\phi(x)\,\mathrm{d}x\right| \leq \left\|\rho^{-1}\phi\right\|_\infty K^{m-1}\,\mu(\mathbb{R}^d)^{1+\beta(m-1)}\int_0^{t_2} t^{-\alpha(m-1)}\mathrm{d}t\,. \tag{4.25}$$

Note that the time integral in the r.h.s. is finite since $\alpha(m-1) < 1$ (recall (4.16) for $p_0 = 1$). We proved above that $\{u_\varepsilon\}$ converges pointwise a.e. (up to subsequences) to a function u which satisfies (3.1), (3.2) and (3.3). If we exploit once again the smoothing effect and the conservation of mass, we easily infer that such convergence also takes place in $\sigma(\mathcal{M}(\mathbb{R}^d), C_0(\mathbb{R}^d))$:

$$\lim_{\varepsilon\to 0}\rho u_\varepsilon(t) = \rho u(t) \quad \text{in } \sigma(\mathcal{M}(\mathbb{R}^d), C_0(\mathbb{R}^d))\,, \text{ for a.e. } t > 0\,. \tag{4.26}$$

Using (4.26), the fact that $\mu_\varepsilon \to \mu$ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ and proceeding exactly as we did in the proof of (4.24), we can let $\varepsilon \to 0$ in (4.25) to get

$$\left|\int_{\mathbb{R}^d} U(x,t_2)\phi(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} U^{\mu}(x)\phi(x)\,\mathrm{d}x\right| \leq \left\|\rho^{-1}\phi\right\|_\infty K^{m-1}\,\mu(\mathbb{R}^d)^{1+\beta(m-1)}\int_0^{t_2} t^{-\alpha(m-1)}\mathrm{d}t \tag{4.27}$$

for a.e. $t_2 > 0$ and $\phi \in \mathcal{D}(\mathbb{R}^d)$, where we denote as U the potential of ρu. Note that, passing to the limit in (4.24) for any nonnegative $\phi \in \mathcal{D}(\mathbb{R}^d)$, we deduce in particular that $U(x,t)$ is nonincreasing in t. Moreover, (4.27) implies that $U(t)$ converges to U^μ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, whence

$$\lim_{t\to 0} U(x,t) = U^\mu(x) \quad \text{for a.e. } x \in \mathbb{R}^d\,. \tag{4.28}$$

Letting $\varepsilon \to 0$ in the conservation of mass (4.1) (applied to $u = u_\varepsilon$ and $\mu = \mu_\varepsilon$), by means e.g. of Fatou's Lemma we obtain

$$\|u(t)\|_{1,\rho} \leq \mu(\mathbb{R}^d) \quad \text{for a.e. } t > 0\,. \tag{4.29}$$

Due to the compactness results recalled in Section 2, from (4.29) we infer that (almost) every sequence $t_n \to 0$ admits a subsequence $\{t_{n_k}\}$ such that $\{\rho u(t_{n_k})\}$ converges to a certain positive finite Radon measure ν in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$. Thanks to (4.28) and [28, Theorem 3.8] we have that $U^\nu(x) = U^\mu(x)$ almost everywhere. Alternatively, such identity can be proved by passing to the limit in $\int_{\mathbb{R}^d} U(x,t_{n_k})\phi(x)\,\mathrm{d}x$, recalling that $U(t_{n_k}) \to U^\mu$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ as $k\to\infty$. Theorem 1.12 of [28] then ensures that two positive finite Radon measures whose potentials are equal almost everywhere must coincide. Hence, $\nu=\mu$ and the limit measure does not depend on the particular subsequence, so that

$$\lim_{t\to 0} \rho u(t) = \mu \quad \text{in } \sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))\,.$$

In order to show that convergence also takes place in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$, it is enough to establish that

$$\lim_{t\to 0} \|u(t)\|_{1,\rho} = \mu(\mathbb{R}^d)\,. \tag{4.30}$$

Since $\rho u(t)$ converges to μ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ as $t\to 0$, we know that

$$\mu(\mathbb{R}^d) \le \liminf_{t\to 0} \|u(t)\|_{1,\rho}\,, \tag{4.31}$$

see again Section 2. But (4.31) and (4.29) entail (4.30).

Finally, the validity of the smoothing estimate (3.5) is just a consequence of passing to the limit in (4.15) (applied to u_ε and $p_0=1$) as $\varepsilon\to 0$ (recall that $\{u_\varepsilon\}$ converges pointwise to u).

At the beginning of the proof we required μ to be compactly supported. Otherwise, take a sequence of compactly supported measures $\{\mu_n\}$ converging to μ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ and consider the corresponding sequence of solutions $\{u_n\}$ to (1.1). Estimates (4.18) and (4.19), as well as the conservation of mass and the smoothing effect, are clearly stable as $\varepsilon\to 0$, thus they also hold upon replacing u_ε with u_n and μ_ε with μ_n. Hence, using the same techniques as above, one proves that $\{u_n\}$ converges to a solution u of (1.1) starting from μ. □

4.4. **Existence and uniqueness of initial traces.** In order to prove Theorem 3.3, we need the next preliminary result.

Lemma 4.6. *Let ν be a signed finite Radon measure such that $U^\nu \ge 0$ almost everywhere. Then $\nu(\mathbb{R}^d)\ge 0$.*

Proof. From the assumptions on U^ν and thanks to Fubini's Theorem, there holds

$$\int_{\mathbb{R}^d} \chi_{B_n}(y)\, U^\nu(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \left(I_{2s} * \chi_{B_n}\right)(x)\,\mathrm{d}\nu = k_{d,s} \int_{\mathbb{R}^d} \left(\int_{B_n} |x-y|^{-d+2s}\,\mathrm{d}y \right) \mathrm{d}\nu \ge 0 \quad \forall n\in\mathbb{N}\,. \tag{4.32}$$

Performing the change of variable $z=y/n$, the last inequality in (4.32) reads

$$\int_{\mathbb{R}^d} \left(\int_{B_1} |x/n - z|^{-d+2s}\,\mathrm{d}z \right) \mathrm{d}\nu \ge 0 \quad \forall n\in\mathbb{N}\,. \tag{4.33}$$

It is plain that for *every* $x\in\mathbb{R}^d$ the sequence $\{\int_{B_1} |x/n-z|^{-d+2s}\,\mathrm{d}z\}$ converges to the positive constant $\int_{B_1}|z|^{-d+2s}\,\mathrm{d}z$ and it is dominated by the latter. Passing to the limit as $n\to\infty$ in (4.33), we get the assertion by dominated convergence (recall that ν is finite). □

Proof of Theorem 3.3. Consider a function u satisfying (3.1), (3.2) and (3.3). Monotonicity in time of the associated potential is proved as we did after (4.21): notice that, for such an argument to work, the running assumptions on γ are required. The same proof holds if, instead of $u\in L^\infty(\mathbb{R}^d\times(\tau,\infty))$, u is only supposed to satisfy $\int_{t_1}^{t_2} u^m(\cdot,\tau)\,\mathrm{d}\tau \in L^1_\rho(\mathbb{R}^d)$ for all $t_2>t_1>0$. Existence of an initial trace μ, meant as convergence in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ along subsequences of a given sequence of times tending to $t=0$, follows by compactness, since we are assuming that solutions belong to $L^\infty((0,\infty); L^1_\rho(\mathbb{R}^d))$. Uniqueness of such a trace is established proceeding as we did after (4.28), using the monotonicity of potentials and the results of [28].

We are left with proving that convergence to μ takes places also in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$, namely that $\operatorname{ess\,lim}_{t\to 0}\int_{\mathbb{R}^d} u(x,t)\,\rho(x)\mathrm{d}x = \mu(\mathbb{R}^d)$. By weak* lower semi-continuity, it is then enough to show that $\operatorname{ess\,lim\,sup}_{t\to 0}\int_{\mathbb{R}^d} u(x,t)\,\rho(x)\mathrm{d}x \le \mu(\mathbb{R}^d)$. Let $U(\cdot,t)$ be the potential of $\{\rho(\cdot)u(\cdot,t)\}$. Again, the monotonicity in time of $U(\cdot,t)$ and the first part of the proof ensure that $U^\mu - U(\cdot,t)\ge 0$

almost everywhere. Therefore, Lemma 4.6 applied to the signed finite Radon measure $\mathrm{d}\nu = \mathrm{d}\mu - u(x,t)\rho(x)\mathrm{d}x$ entails $\mu(\mathbb{R}^d) \geq \int_{\mathbb{R}^d} u(x,t)\,\rho(x)\mathrm{d}x$. Letting $t \to 0$ concludes the proof.

□

4.5. **Strong solutions and decrease of the norms.** In order to justify rigorously some of the above computations, it is essential to show that the weak solutions constructed in Lemma 4.2 are strong. By a "strong solution", following [18, Section 6.2], we mean a weak solution u such that $u_t \in L^\infty((\tau,\infty), L^1_\rho(\mathbb{R}^d))$ for all $\tau > 0$. The fact that our solutions are indeed strong can be proved as in [18, Section 8.1]. The first step consists in showing that $\rho(\cdot)u_t(\cdot,t)$ is a finite Radon measure which satisfies the estimate

$$\|\rho\, u_t(t)\|_{\mathcal{M}(\mathbb{R}^d)} \leq \frac{2\,\|u_0\|_{1,\rho}}{(m-1)t} \quad \forall t > 0\,, \tag{4.34}$$

where now, by $\mathcal{M}(\mathbb{R}^d)$ we mean the Banach space of *signed* finite Radon measures on $\mathbb{R}^d$, equipped with the usual norm of the variation. As in [43, Lemma 8.5], this follows by using the inequality

$$\int_{\mathbb{R}^d} [u(x,t) - \tilde{u}(x,t)]_+ \,\rho(x)\mathrm{d}x \leq \int_{\mathbb{R}^d} [u_0(x) - \tilde{u}_0(x)]_+ \,\rho(x)\mathrm{d}x \quad \forall t > 0\,, \tag{4.35}$$

where u and $\tilde{u}$ are the solutions to (4.3) *constructed in Lemma 4.2* corresponding to the initial data u_0 and $\tilde{u}_0$, respectively. Such inequality does hold for the approximate solutions u_η and $\tilde{u}_\eta$ used in the proof of Lemma 4.2 (see [38, Proposition 3.4]), whence (4.35) follows by passing to the limit. Afterwards, as [18, Lemma 8.1], one proves that $z := u^{\frac{m+1}{2}}$ fulfills (3.7). In particular,

$$z_t \in L^2_{\mathrm{loc}}((0,\infty); L^2_\rho(\mathbb{R}^d))\,. \tag{4.36}$$

Thanks to (4.34) and (4.36), the abstract result contained in [6, Theorem 1.1] ensures that $u_t \in L^1_{\mathrm{loc}}((0,\infty); L^1_\rho(\mathbb{R}^d))$. In particular, (4.34) holds with $\|\rho\, u_t(t)\|_{\mathcal{M}(\mathbb{R}^d)}$ replaced by $\|u_t(t)\|_{1,\rho}$, whence the assertion.

An important consequence of the fact that the solutions constructed in Lemma 4.2 are strong is the *decrease of their L^p_ρ norms* for any $p \in [1,\infty]$. Indeed, by definition of strong solution, for any $p \in (1,\infty)$, we are allowed to multiply the differential equation in (4.3) by u^{p-1} and integrate in $\mathbb{R}^d \times [t_1,t_2]$. By Stroock-Varopoulos inequality (4.9) (let $v = u^m$ and $q = (p+m-1)/m$), we get

$$\int_{\mathbb{R}^d} u^p(x,t_2)\,\rho(x)\mathrm{d}x - \int_{\mathbb{R}^d} u^p(x,t_1)\,\rho(x)\mathrm{d}x = -p\int_{t_1}^{t_2}\int_{\mathbb{R}^d} u^{p-1}(x,t)(-\Delta)^s(u^m)(x,t)\,\mathrm{d}x\mathrm{d}t \leq 0 \tag{4.37}$$

for all $t_2 > t_1 > 0$. The validity of (4.37) down to $t_1 = 0$ follows by using the approximate solutions $\{u_\eta\}$ from the proof of Lemma 4.2 and letting $\eta \to 0$. The case $p = \infty$ can be handled by approximation.

5. Uniqueness of weak solutions

Prior to the proof of Theorem 3.4, we need some technical lemmas. Hereafter, by "weak solution" to (1.1), we shall mean a solution in the sense of Definition 3.1.

Lemma 5.1. *Let $d > 2s$ and assume that ρ satisfies* (1.2) *for some $\gamma \in [0,2s) \cap [0,d-2s]$ and $\gamma_0 \in [0,\gamma]$. Let u be a weak solution to* (1.1)*. Then the potential $U(\cdot,t)$ of $\rho(\cdot)u(\cdot,t)$ admits an absolutely continuous version (in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$) which is nonincreasing in t.*

Proof. One proceeds as in the first part of the proof of Theorem 3.2, using the same techniques we exploited to prove (4.21) rigorously. □

Lemma 5.2. *Let $d > 2s$ and assume that ρ satisfies* (1.2) *for some $\gamma \in [0,2s) \cap [0,d-2s]$ and $\gamma_0 \in [0,\gamma]$. Let u be a weak solution to* (1.1)*, taking the initial datum μ whose potential is U^μ. Then there holds*

$$\lim_{t\downarrow 0} U(x,t) = U^\mu(x) \quad \forall x \in \mathbb{R}^d\,. \tag{5.1}$$

Proof. It is a direct application of Theorem 3.9 of [28] but, for the reader's convenience, we give some details.

Thanks to Theorem 3.8 of [28] and to the monotonicity ensured by Lemma 5.1, we have that the limit in (5.1) is taken at least for a.e. $x \in \mathbb{R}^d$. However, for what follows it will be crucial to prove that it is taken *for every* $x \in \mathbb{R}^d$. To this end we make use again of the monotonicity property provided by Lemma 5.1. In fact, Lemma 1.12 of [28] shows that, as a consequence of the monotonicity of potentials, there exists a positive finite Radon measure ν, whose potential is denoted by U^ν, and a constant $A \geq 0$ such that

$$\lim_{t \downarrow 0} U(x,t) = U^\nu(x) + A \quad \forall x \in \mathbb{R}^d \,.$$

Since (5.1) holds almost everywhere,

$$U^\mu(x) = U^\nu(x) + A \quad \text{for a.e. } x \in \mathbb{R}^d \,. \tag{5.2}$$

But using the corollary at page 129 of [28], from (5.2) we deduce that necessarily $A = 0$. Hence, (5.2) implies that $U^\nu = U^\mu$ almost everywhere, and from Theorem 1.12 of [28] we know that two potentials coinciding a.e. in fact coincide everywhere, whence (5.1) follows. □

5.1. **Main ideas in the proof of uniqueness.** Since the proof of Theorem 3.4 is rather delicate, we point out its main ingredients. We should note that from a general viewpoint it is based on a "duality method", and in particular it is modeled on the uniqueness proof given by M. Pierre in [34]. We comment again that our uniqueness result seems to be new even if $s = 1$, in the weighted case, or if $\rho \equiv 1$ when $s \in (0,1)$.

Let u_1 and u_2 be two weak solutions to (1.1) such that they both take a common positive, finite Radon measure μ as initial datum. We denote as $U_1(\cdot,t)$ and $U_2(\cdot,t)$ the potentials of $\rho(\cdot)u_1(\cdot,t)$ and $\rho(\cdot)u_2(\cdot,t)$, respectively. Fix once for all the parameters $h, T > 0$ and consider the function

$$g(x,t) := U_2(x,t+h) - U_1(x,t) \quad \forall (x,t) \in \mathbb{R}^d \times (0,T] \,. \tag{5.3}$$

Proceeding again as in the proof of Theorem 3.2 (under the hypothesis $\gamma \leq d - 2s$, see the proof of (4.21)), we get that $g(\cdot,t)$ is an absolutely continuous curve (for instance in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$) satisfying

$$\rho(x) g_t(x,t) = \rho(x) \left(u_1^m(x,t) - u_2^m(x,t+h) \right) = -a(x,t)(-\Delta)^s(g)(x,t) \tag{5.4}$$

for a.e. $(x,t) \in \mathbb{R}^d \times (0,T)$, where we define the function a as

$$a(x,t) := \begin{cases} \frac{u_1^m(x,t) - u_2^m(x,t+h)}{u_1(x,t) - u_2(x,t+h)} & \text{if } u_1(x,t) \neq u_2(x,t+h) \,, \\ 0 & \text{if } u_1(x,t) = u_2(x,t+h) \,, \end{cases} \tag{5.5}$$

and we used the fact that, thanks to the properties of Riesz potentials,

$$(-\Delta)^s(g)(x,t) = \rho(x) u_2(x,t+h) - \rho(x) u_1(x,t) \,.$$

Note that, since $m > 1$ and $u_1, u_2 \in L^\infty(\mathbb{R}^d \times (\tau,\infty))$ for all $\tau > 0$, a is a nonnegative function belonging to $L^\infty(\mathbb{R}^d \times (\tau,\infty))$ for all $\tau > 0$.

Hence g is a solution to the *linear fractional* equation (5.4). Moreover, by Lemmas 5.1 and 5.2, $g(x,0) \leq 0$ for a.e. $x \in \mathbb{R}^d$. If we could apply the maximum principle, then we would get $g \leq 0$ in $\mathbb{R}^d \times (0,\infty)$. This would imply $u_1 \leq u_2$ and, by interchanging the roles of u_1 and u_2, $u_1 = u_2$. However, a priori a maximum principle is not available for solutions to (5.4). We then consider the "dual" problem

$$\begin{cases} \rho(x)\varphi_t = (-\Delta)^s(a\varphi) & \text{in } \mathbb{R}^d \times (0,T) \,, \\ \varphi(x,T) = \psi(x) & \text{on } \mathbb{R}^d \times \{T\} \,, \end{cases}$$

for any $\psi \in \mathcal{D}_+(\mathbb{R}^d)$. Suppose for a moment that it admits a unique smooth solution φ. Multiplying (5.4) by φ and integrating by parts we formally obtain

$$\int_{\mathbb{R}^d} g(x,T)\rho(x)\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x,0)g(x,0)\,\mathrm{d}x. \tag{5.6}$$

The conclusion would again follow should a maximum principle for (5.6) hold, and in order to justify rigorously its applicability a further approximation is necessary. In fact, for every $n \in \mathbb{N}$ and $\varepsilon > 0$, we consider a family $\{\psi_{n,\varepsilon}\}$ which solves, in a sense that will be clarified later, the problem

$$\begin{cases} \rho(x)\left(\psi_{n,\varepsilon}\right)_t = (-\Delta)^s\left[\left(a_n+\varepsilon\right)\psi_{n,\varepsilon}\right] & \text{in } \mathbb{R}^d \times (0,T)\,, \\ \psi_{n,\varepsilon} = \psi & \text{on } \mathbb{R}^d \times \{T\}\,, \end{cases} \tag{5.7}$$

where $\psi \in \mathcal{D}_+(\mathbb{R}^d)$. The sequence $\{a_n\}$ is a suitable approximation of the function a defined in (5.5). In particular we suppose that, for every $n \in \mathbb{N}$, $a_n(x,t)$ is a piecewise constant function of t (regular in x) on the time intervals $(T-(k+1)T/n, T-kT/n]$, for any $k \in \{0,\ldots,n-1\}$. Thanks to Theorem 3.7 and to Proposition B.1 below, we are then able to treat problem (5.7) by means of standard semigroup theory. Here the Markov property for the linear semigroup associated to the operator $A = \rho^{-1}(-\Delta)^s$ will have a crucial role. Let us mention that in [34, Theorem 1], where $s=1$, $\rho \equiv 1$, in view of standard parabolic theory it was not necessary to approximate the function a by a piecewise constant function of t. Using the family $\{\psi_{n,\varepsilon}\}$ and passing to the limit as $n \to \infty$ and then as $\varepsilon \to 0$ we get the next crucial identity:

$$\int_{\mathbb{R}^d} g(x,T)\psi(x)\,\rho(x)\mathrm{d}x = \int_{\mathbb{R}^d} g(x,t)\,\mathrm{d}\nu(t) \quad \text{for a.e. } t \in (0,T)\,, \tag{5.8}$$

where $\{\nu(t)\}$ is a specific family of positive finite Radon measures. More precisely, $\nu(t)$ is the limit in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ as $\varepsilon \to 0$ of $\{\rho(\cdot)\psi_\varepsilon(\cdot,t)\}$, where ψ_ε is in turn the weak limit in $L^2_\rho(\mathbb{R}^d \times (\tau,T))$ (for all $\tau \in (0,T)$) as $n \to \infty$ of $\{\psi_{n,\varepsilon}\}$. Note that, roughly speaking, (5.8) corresponds to identity (5.6) in the previous formal argument. Finally, we prove rigorously that the r.h.s. of (5.8) has a nonpositive limit as $t \to 0$, whence the conclusion follows.

5.2. **Construction and properties of the family $\{\psi_{n,\varepsilon}\}$.** We begin our proof by introducing the functions $\psi_{n,\varepsilon}$, which formally solve (5.7).

Lemma 5.3. *Let $d > 2s$ and assume that ρ satisfies* (1.2) *for some $\gamma \in (0,2s)$. Let $\{a_n\}$ be a sequence of functions converging a.e. to the function a as in* (5.5) *such that:*

- *for any $n \in \mathbb{N}$ and $t > 0$, $a_n(x,t)$ is a regular function of x;*
- *for any $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$, $a_n(x,t)$ is a piecewise constant function of t on the time intervals $(T-(k+1)T/n, T-kT/n]$, for any $k \in \{0,\ldots,n-1\}$;*
- *$\{\|a_n\|_{L^\infty(\mathbb{R}^d\times(\tau,\infty))}\}$ is uniformly bounded in n for any $\tau > 0$.*

Then, for any $\varepsilon > 0$ and any $\psi \in \mathcal{D}_+(\mathbb{R}^d)$, there exists a nonnegative solution $\psi_{n,\varepsilon}$ to problem (5.7)*, in the sense that $\psi_{n,\varepsilon}(\cdot,t)$ is a continuous curve in $L^p_\rho(\mathbb{R}^d)$ (for all $p \in (1,\infty)$) satisfying $\psi_{n,\varepsilon}(\cdot,0) = \psi(\cdot,0)$ and it is absolutely continuous on $(T-(k+1)T/n, T-kT/n)$ for all $k \in \{0,\ldots,n-1\}$, so that the identity*

$$\psi_{n,\varepsilon}(\cdot,t_2) - \psi_{n,\varepsilon}(\cdot,t_1) = \int_{t_1}^{t_2} \rho^{-1}(\cdot)(-\Delta)^s\left[\left(a_n+\varepsilon\right)\psi_{n,\varepsilon}\right](\cdot,\tau)\,\mathrm{d}\tau \tag{5.9}$$

$$\forall t_1,t_2 \in \left(T - \frac{(k+1)T}{n}, T - \frac{kT}{n}\right),\ \forall k \in \{0,\ldots,n-1\}$$

holds in $L^p_\rho(\mathbb{R}^d)$ for all $p \in (1,\infty)$. Moreover,

$$\psi_{n,\varepsilon} \in L^\infty((0,T); L^p_\rho(\mathbb{R}^d)) \quad \forall p \in [1,\infty] \quad \text{and} \quad \|\psi_{n,\varepsilon}(t)\|_{1,\rho} \le \|\psi\|_{1,\rho} \quad \forall t \in [0,T]\,. \tag{5.10}$$

Proof. To construct $\psi_{n,\varepsilon}$ as in the statement, we first define ζ_1 as the solution of

$$\begin{cases} \rho(x)\left(\zeta_1\right)_t = (-\Delta)^s\left[\left(a_n(T)+\varepsilon\right)\zeta_1\right] & \text{in } \mathbb{R}^d \times \left(T-\frac{T}{n}, T\right)\,, \\ \zeta_1 = \psi & \text{on } \mathbb{R}^d \times \{T\}\,. \end{cases} \tag{5.11}$$

To construct such a solution, one can for instance exploit the change of variable

$$\phi_1(x,t) := \left(a_n\left(x,T\right)+\varepsilon\right)\zeta_1(x,t)\,, \tag{5.12}$$

where ϕ_1 is the solution of

$$\begin{cases} (\phi_1)_t = (a_n(T)+\varepsilon)\,\rho^{-1}(-\Delta)^s(\phi_1) & \text{in } \mathbb{R}^d \times \left(T-\frac{T}{n}, T\right), \\ \phi_1 = (a_n(T)+\varepsilon)\,\psi & \text{on } \mathbb{R}^d \times \{T\}. \end{cases} \tag{5.13}$$

Problem (5.13) is indeed solvable by standard semigroup theory. In fact, consider the operator $A_1 := \rho_1^{-1}(-\Delta)^s$, where we have set $\rho_1(x) := (a_n(x,T)+\varepsilon)^{-1}\rho(x)$, with domain $X_{s,\rho_1} = X_{s,\rho}$ (see Definition 3.6). A_1 is positive, self-adjoint and generates a Markov semigroup on $L^2_{\rho_1}(\mathbb{R}^d)$. These properties follow from Theorem 3.7. Our initial datum ϕ_1 belongs to $L^p_{\rho_1}(\mathbb{R}^d)$ for all $p \in [1,\infty]$, and this is enough in order to have a solution to (5.13) which is continuous up to $t = T$ and absolutely continuous in $\left(T-\frac{T}{n}, T\right)$ in $L^p_{\rho_1}(\mathbb{R}^d)$ for all $p \in (1,\infty)$. In fact, the semigroup associated with A_1 enjoys the Markov property and therefore, as a consequence of [13, Theorems 1.4.1, 1.4.2], can be extended to a contraction semigroup on $L^p_{\rho_1}(\mathbb{R}^d)$ (consistent with the original semigroup on $L^2_{\rho_1}(\mathbb{R}^d)\cap L^p_{\rho_1}(\mathbb{R}^d)$) for all $p\in[1,\infty]$, which is analytic with a suitable angle $\theta_p > 0$ if $p \in (1,\infty)$. By classical results (see e.g. [33, Theorem 5.2 at p. 61]) the latter property ensures in particular that problem (5.13) is solved by a *differentiable* curve $\phi_1(\cdot,t)$ in $L^p_{\rho_1}(\mathbb{R}^d)$ for all $p \in (1,\infty)$. Going back to the original variable ζ_1 through (5.12), we deduce that it solves (5.11) in the same sense in which ϕ_1 solves (5.13). Having at our disposal such a ζ_1, we can then solve the problem

$$\begin{cases} \rho(x)\,(\zeta_2)_t = (-\Delta)^s\left[\left(a_n\left(T-\frac{T}{n}\right)+\varepsilon\right)\zeta_2\right] & \text{in } \mathbb{R}^d \times \left(T-\frac{2T}{n}, T-\frac{T}{n}\right), \\ \zeta_2 = (a_n(x,T)+\varepsilon)^{-1}\,\phi_1 & \text{on } \mathbb{R}^d \times \left\{T-\frac{T}{n}\right\}, \end{cases}$$

just by proceeding as above. That is, we perform the change of variable

$$\phi_2(x,t) := \left(a_n\left(x, T-\frac{T}{n}\right)+\varepsilon\right)\zeta_2(x,t)$$

and take ϕ_2 as the solution of

$$\begin{cases} (\phi_2)_t = \left(a_n(T-\frac{T}{n})+\varepsilon\right)\rho^{-1}(-\Delta)^s(\phi_2) & \text{in } \mathbb{R}^d \times \left(T-\frac{2T}{n}, T-\frac{T}{n}\right), \\ \phi_2 = \left(a_n(T-\frac{T}{n})+\varepsilon\right)\zeta_1 = \frac{\left(a_n(T-\frac{T}{n})+\varepsilon\right)}{(a_n(T)+\varepsilon)}\phi_1 & \text{on } \mathbb{R}^d \times \left\{T-\frac{T}{n}\right\}. \end{cases}$$

It is clear how the procedure goes on and allows us to obtain a solution $\psi_{n,\varepsilon}$ to (5.7) in the sense of the statement, just by defining it as

$$\psi_{n,\varepsilon}(\cdot,t) := \zeta_{k+1}(\cdot,t) \quad \forall t \in \left(T-\frac{(k+1)T}{n}, T-\frac{kT}{n}\right], \quad \forall k \in \{0,\ldots,n-1\}.$$

Finally, since

$$\rho_{k+1}^{-1}(-\Delta)^s$$

generates a contraction semigroup on $L^p_{\rho_{k+1}}(\mathbb{R}^d)$ for all $p \in [1,\infty]$, where

$$\rho_{k+1}(x) := \left(a_n\left(x, T-\frac{kT}{n}\right)+\varepsilon\right)^{-1}\rho(x), \tag{5.14}$$

the inequalities

$$\|\phi_{k+1}(t)\|_{p,\rho_{k+1}} \le \left\| \frac{\left(a_n(T-\frac{kT}{n})+\varepsilon\right)}{\left(a_n(T-\frac{(k-1)T}{n})+\varepsilon\right)}\phi_k\left(T-\frac{kT}{n}\right)\right\|_{p,\rho_{k+1}} \tag{5.15}$$
$$\forall t \in \left(T-\frac{(k+1)T}{n}, T-\frac{kT}{n}\right], \ \forall p \in [1,\infty]$$

hold for any $k \in \{0,\ldots,n-1\}$ (on the r.h.s. of (5.15) for $k=0$ we conventionally set $\phi_0 = \psi$ and $a_n(T+T/n)+\varepsilon = 1$). Going back to the variables ζ_{k+1} and recalling (5.14), from (5.15) one deduces (5.10): in fact, for $p=1$ it is easy to see that the terms containing a_n cancel out and give the corresponding inequality, while for $p>1$ such terms remain and one obtains an inequality of the type of $\|\psi_{n,\varepsilon}(t)\|_{p,\rho} \le C(n,\varepsilon)\|\psi\|_{p,\rho}$, where $C(n,\varepsilon)$ is a positive constant depending on n,ε. □

Lemma 5.4. *Let $d > 2s$ and assume that ρ satisfies* (1.2) *for some $\gamma \in (0, 2s) \cap (0, d-2s]$. Let g be as in* (5.3)*, a as in* (5.5) *and a_n, $\psi_{n,\varepsilon}$, ψ as in Lemma 5.3. Then the identity*

$$\begin{aligned}&\int_{\mathbb{R}^d} g(x,T)\psi(x)\,\rho(x)\mathrm{d}x - \int_{\mathbb{R}^d} g(x,t)\psi_{n,\varepsilon}(x,t)\,\rho(x)\mathrm{d}x \\ &= \int_t^T \int_{\mathbb{R}^d} \left(a_n(x,\tau) + \varepsilon - a(x,\tau)\right)(-\Delta)^s(g)(x,\tau)\,\psi_{n,\varepsilon}(x,\tau)\,\mathrm{d}x\mathrm{d}\tau\end{aligned} \tag{5.16}$$

holds for all $t \in (0, T]$.

Proof. To begin with, let us set $t_k := T(n-k)/n$ for all $k \in \{0, \dots, n\}$. Recall that, from Lemma 5.3, $\psi_{n,\varepsilon}(\cdot, t)$ is a continuous curve in $L^p_\rho(\mathbb{R}^d)$ on $(0, T]$, absolutely continuous on any interval (t_{k+1}, t_k) for $k \in \{0, \dots, n-1\}$ and satisfying the differential equation in (5.7) on such intervals, for all $p \in (1, \infty)$. Moreover, $g(\cdot, t)$ is an absolutely continuous curve in $L^p_\rho(\mathbb{R}^d)$ on $(0, T]$ for all p such that

$$p \in \left(\frac{d-\gamma}{d-2s}, \infty\right). \tag{5.17}$$

Since $g(x,t)$ is a continuous function of x (recall Lemma A.6) and the weight $\rho(x)$ is locally integrable, in order to prove that $g(\cdot,t) \in L^p_\rho(\mathbb{R}^d)$ for all p as in (5.17) it suffices to show that $g(\cdot, t) \in L^p_\rho(B_1^c)$. To this end, still Lemma A.6 ensures that $g(\cdot, t) \in L^p(\mathbb{R}^d)$ for all p satisfying (A.11): the latter property and Hölder's inequality imply that $g(\cdot, t) \in L^p_\rho(B_1^c)$ for all p as in (5.17).

The fact that $g(\cdot, t)$ is also absolutely continuous in $L^p_\rho(\mathbb{R}^d)$ on the time interval $(0, T]$ is a consequence of (5.4) and of the integrability properties of u_1, u_2. Hence, due to Lemma 5.3, we get that

$$t \mapsto \int_{\mathbb{R}^d} g(x,t)\psi_{n,\varepsilon}(x,t)\,\rho(x)\mathrm{d}x \tag{5.18}$$

is a continuous function on $(0, T]$, absolutely continuous on each interval (t_{k+1}, t_k) and satisfies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} g(x,t)\psi_{n,\varepsilon}(x,t)\,\rho(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \left\{-a(x,t)(-\Delta)^s(g)(x,t)\,\psi_{n,\varepsilon}(x,t) + g(x,t)\,(-\Delta)^s\left[(a_n+\varepsilon)\,\psi_{n,\varepsilon}\right](x,t)\right\}\mathrm{d}x\end{aligned} \tag{5.19}$$

there. As we have just seen, $g(\cdot, t) \in L^p_\rho(\mathbb{R}^d)$ for all p satisfying (5.17) and $\rho^{-1}(\cdot)(-\Delta)^s(g)(\cdot, t) \in L^p_\rho(\mathbb{R}^d)$ for all $p \in [1, \infty]$. Moreover, as a consequence of Lemma 5.3, we have that $(a_n(\cdot,t) + \varepsilon)\psi_{n,\varepsilon}(\cdot, t) \in L^p_\rho(\mathbb{R}^d)$ for all $p \in [1, \infty]$ and $\rho^{-1}(\cdot)(-\Delta)^s[(a_n(\cdot,t)+\varepsilon)\psi_{n,\varepsilon}(\cdot,t)] \in L^p_\rho(\mathbb{R}^d)$ for all $p \in (1, \infty)$. We are therefore in position to apply Proposition B.1 to the r.h.s. of (5.19) (note that the interval $((d-\gamma)/(d-2s), \infty) \cap [2, 2(d-\gamma)/(d-2s))$ is not empty) to get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} g(x,t)\psi_{n,\varepsilon}(x,t)\,\rho(x)\mathrm{d}x = \int_{\mathbb{R}^d} \left(a_n(x,t)+\varepsilon - a(x,t)\right)(-\Delta)^s(g)(x,t)\,\psi_{n,\varepsilon}(x,t)\,\mathrm{d}x\,. \tag{5.20}$$

But the r.h.s. of (5.20) is in $L^1((\tau, T))$ for any $\tau \in (0, T)$, from which (5.18) is absolutely continuous on the whole of $(0, T]$ and not only on (t_{k+1}, t_k). Integrating (5.20) between t and T then yields (5.16). □

Now we prove a key "conservation of mass" property for $\psi_{n,\varepsilon}$.

Lemma 5.5. *Let $d > 2s$ and assume that ρ satisfies* (1.2) *for some $\gamma \in (0, 2s)$. Let $\psi_{n,\varepsilon}$ and ψ be as in Lemma 5.3. Then the L^1_ρ norm of $\psi_{n,\varepsilon}(\cdot, t)$ is preserved, that is*

$$\int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x,t)\,\rho(x)\mathrm{d}x = \int_{\mathbb{R}^d} \psi(x)\,\rho(x)\mathrm{d}x \quad \forall t \in (0, T]\,. \tag{5.21}$$

Proof. Multiplying (5.9) by any $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and integrating in $\mathbb{R}^d$, we obtain:

$$\begin{aligned}&\int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x,t^*)\varphi(x)\,\rho(x)\mathrm{d}x - \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x,t_*)\varphi(x)\,\rho(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^d} (-\Delta)^s(\varphi)(x) \left(\int_{t_*}^{t^*} (a_n(x,\tau)+\varepsilon)\,\psi_{n,\varepsilon}(x,\tau)\,\mathrm{d}\tau \right) \mathrm{d}x\end{aligned} \tag{5.22}$$

for all $t_*, t^* \in (t_{k+1}, t_k)$. Since the L^1_ρ norm of $\psi_{n,\varepsilon}(\cdot,t)$ is bounded by the L^1_ρ norm of the final datum ψ (recall (5.10)), from (5.22) we get:

$$\left| \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x,t^*)\varphi(x)\,\rho(x)\mathrm{d}x - \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x,t_*)\varphi(x)\,\rho(x)\mathrm{d}x \right| \le \underline{C}\,|t^*-t_*|\,\|\psi\|_{1,\rho} \left\|\rho^{-1}(-\Delta)^s(\varphi)\right\|_\infty , \tag{5.23}$$

where $\underline{C} := \|a_n+\varepsilon\|_{L^\infty(\mathbb{R}^d\times(t_*\wedge t^*,T))}$ is a positive constant independent of n and ε. Replacing φ with the cut-off function ξ_R (defined in Lemma A.3) and estimating the r.h.s. of (5.23) as in the proof of Proposition 4.1 yields

$$\begin{aligned}&\left| \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x,t^*)\xi_R(x)\rho(x)\mathrm{d}x - \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x,t_*)\xi_R(x)\rho(x)\mathrm{d}x \right| \\ \le& \underline{C}\,|t^*-t_*|\,\|\psi\|_{1,\rho}\, c^{-1} \left(R^{-2s}+R^{-2s+\gamma}\right) \|(1+|x|^\gamma)(-\Delta)^s(\xi)\|_\infty\end{aligned} \tag{5.24}$$

for all $R>0$ and $t_*, t^* \in (t_{k+1}, t_k)$, c being as in (1.2). Recalling that $\psi_{n,\varepsilon}(\cdot,t)$ is a continuous curve (for instance in $L^2_\rho(\mathbb{R}^d)$) on $(0,T]$, we can extend the validity of (5.24) (and (5.23)) to any $t_*, t^* \in (0,T]$. By choosing $t^*=T$ and letting $R\to\infty$ in (5.24) we finally get (5.21). □

In the next lemma we introduce the Riesz potential of $\rho(\cdot)\psi_{n,\varepsilon}(\cdot,t)$, which will play a fundamental role below.

Lemma 5.6. *Let $d>2s$ and assume that ρ satisfies (1.2) for some $\gamma\in(0,2s)\cap(0,d-2s]$. Let a_n, $\psi_{n,\varepsilon}$ and ψ be as in Lemma 5.3. We denote as $H_{n,\varepsilon}(\cdot,t)$ the Riesz potential of $\rho(\cdot)\psi_{n,\varepsilon}(\cdot,t)$, that is*

$$H_{n,\varepsilon}(x,t) := [I_{2s} * (\rho(\cdot)\psi_{n,\varepsilon}(\cdot,t))](x) \quad \forall (x,t)\in\mathbb{R}^d\times(0,T]\,.$$

Then $H_{n,\varepsilon}(\cdot,t)\in\dot{H}^s(\mathbb{R}^d)$ and the identity

$$\|I_{2s}*(\rho\psi)\|^2_{\dot{H}^s} = \|H_{n,\varepsilon}(t)\|^2_{\dot{H}^s} + 2\int_t^T\int_{\mathbb{R}^d} (a_n(x,\tau)+\varepsilon)\,\psi^2_{n,\varepsilon}(x,\tau)\,\rho(x)\mathrm{d}x\mathrm{d}\tau \tag{5.25}$$

holds for all $t\in(0,T]$.

Proof. First notice that $\rho^{-1}(\cdot)(-\Delta)^s(H_{n,\varepsilon})(\cdot,t) = \psi_{n,\varepsilon}(\cdot,t)\in L^p_\rho(\mathbb{R}^d)$ for all $p\in[1,\infty]$ (recall (5.10)) and $H_{n,\varepsilon}(\cdot,t)\in L^p_\rho(\mathbb{R}^d)$ for all p satisfying (5.17) (this can be proved by exploiting Lemma A.6 exactly as in the proof of Lemma 5.4). Again, since the interval $((d-\gamma)/(d-2s),\infty)\cap[2,2(d-\gamma)/(d-2s))$ is not empty, applying Proposition B.1 we get that $H_{n,\varepsilon}(\cdot,t)\in\dot{H}^s(\mathbb{R}^d)$ and the identity

$$\|H_{n,\varepsilon}(t)\|^2_{\dot{H}^s} = \int_{\mathbb{R}^d} H_{n,\varepsilon}(x,t)\,(-\Delta)^s\,(H_{n,\varepsilon})\,(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^d} H_{n,\varepsilon}(x,t)\psi_{n,\varepsilon}(x,t)\,\rho(x)\mathrm{d}x \tag{5.26}$$

holds. Thanks to the validity of the differential equation

$$(H_{n,\varepsilon})_t\,(x,t) = (a_n(x,t)+\varepsilon)\,\psi_{n,\varepsilon}(x,t) \quad \text{for a.e. } (x,t)\in\mathbb{R}^d\times(0,T)\,, \tag{5.27}$$

which can be justified as we did for (5.4), taking the time derivative of (5.26) in the intervals (t_{k+1},t_k), using (5.27), (5.7) and again Proposition B.1, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|H_{n,\varepsilon}(t)\|^2_{\dot{H}^s} = 2\int_{\mathbb{R}^d} (a_n(x,t)+\varepsilon)\,\psi^2_{n,\varepsilon}(x,t)\,\rho(x)\mathrm{d}x\,. \tag{5.28}$$

A priori, from (5.26), we have that $\|H_{n,\varepsilon}(t)\|^2_{\dot{H}^s}$ is continuous on $(0,T]$ and absolutely continuous only on (t_{k+1},t_k). However, the r.h.s. of (5.28) is in $L^1((\tau,T))$ for any $\tau\in(0,T)$. Hence, (5.25) just follows by integrating (5.28) from t to T. □

5.3. **Passing to the limit as $n \to \infty$.** The goal of the next lemma is to show that, as $n \to \infty$, $\{\psi_{n,\varepsilon}\}$ suitably converges to a limit function ψ_ε that enjoys some crucial properties.

Lemma 5.7. *Let $d > 2s$ and assume that ρ satisfies* (1.2) *for some $\gamma \in (0, 2s) \cap (0, d-2s]$. Let u_1 and u_2 be two weak solutions to problem* (1.1)*, taking the common positive finite Radon measure μ as initial datum. Let g be as in* (5.3)*, a as in* (5.5) *and $\psi_{n,\varepsilon}, \psi$ as in Lemma 5.3. Then, up to subsequences, $\{\psi_{n,\varepsilon}\}$ converges weakly in $L^2_\rho(\mathbb{R}^d \times (\tau, T))$ (for all $\tau \in (0,T)$) to a suitable nonnegative function ψ_ε and $\{\rho(\cdot)\psi_{n,\varepsilon}(\cdot,t)\}$ converges to $\rho(\cdot)\psi_\varepsilon(\cdot,t)$ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ for a.e. $t \in (0,T)$. Moreover, ψ_ε enjoys the following properties:*

$$\int_{\mathbb{R}^d} \psi_\varepsilon(x,t)\, \rho(x)\mathrm{d}x = \int_{\mathbb{R}^d} \psi(x)\, \rho(x)\mathrm{d}x\,, \tag{5.29}$$

$$\begin{aligned}&\int_{\mathbb{R}^d} \psi(x)\varphi(x)\, \rho(x)\mathrm{d}x - \int_{\mathbb{R}^d} \psi_\varepsilon(x,t)\varphi(x)\, \rho(x)\mathrm{d}x \\ =&\int_{\mathbb{R}^d} (-\Delta)^s(\varphi)(x) \left(\int_t^T (a(x,\tau)+\varepsilon)\, \psi_\varepsilon(x,\tau)\,\mathrm{d}\tau \right) \mathrm{d}x\,,\end{aligned} \tag{5.30}$$

$$\begin{aligned}&\left| \int_{\mathbb{R}^d} g(x,T)\psi(x)\, \rho(x)\mathrm{d}x - \int_{\mathbb{R}^d} g(x,t)\psi_\varepsilon(x,t)\, \rho(x)\mathrm{d}x \right| \\ \le& \varepsilon\,(T-t)\, \|\psi\|_{1,\rho} \, \|u_2(\tau+h) - u_1(\tau)\|_{L^\infty(\mathbb{R}^d \times (t,T))}\end{aligned} \tag{5.31}$$

for a.e. $t \in (0,T)$, for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

Proof. From (5.25) one gets that, up to subsequences, $\{\psi_{n,\varepsilon}\}$ converges weakly in $L^2_\rho\left(\mathbb{R}^d \times (\tau,T)\right)$ (for all $\tau \in (0,T)$) to a suitable ψ_ε. Moreover, thanks to the uniform boundedness of $\{\rho(\cdot)\psi_{n,\varepsilon}(\cdot,t)\}$ in $L^1(\mathbb{R}^d)$ (see (5.10)), for every $t \in (0,T)$ there exists a subsequence (which a priori may depend on t) such that $\{\rho(\cdot)\psi_{n,\varepsilon}(\cdot,t)\}$ converges in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ to some positive, finite Radon measure $\nu(t)$ (recall the preliminary results of Section 2). We aim at identifying (at least for almost every $t \in (0,T)$) $\nu(t)$ with $\rho(\cdot)\psi_\varepsilon(\cdot,t)$, so that a posteriori the subsequence does not depend on t. In order to do that, let $t \in (0,T)$ be a Lebesgue point of $\psi_\varepsilon(\cdot,t)$ (as a curve in $L^1((\tau,T); L^2_\rho(\mathbb{R}^d))$. Taking any $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and using (5.23), we obtain:

$$\begin{aligned}&\left| \int_t^{t+\delta} \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x,\tau)\varphi(x)\, \rho(x)\mathrm{d}x\mathrm{d}\tau - \int_t^{t+\delta} \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x,t)\varphi(x)\, \rho(x)\mathrm{d}x\mathrm{d}\tau \right| \\ \le& \int_t^{t+\delta} \left| \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x,\tau)\varphi(x)\, \rho(x)\mathrm{d}x - \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x,t)\varphi(x)\, \rho(x)\mathrm{d}x \right| \mathrm{d}\tau \\ \le& \int_t^{t+\delta} \underline{C}\,(\tau - t)\, \|\psi\|_{1,\rho} \left\| \rho^{-1}(-\Delta)^s(\varphi) \right\|_\infty \mathrm{d}\tau = \frac{\delta^2}{2} \underline{C}\, \|\psi\|_{1,\rho} \left\| \rho^{-1}(-\Delta)^s(\varphi) \right\|_\infty\end{aligned} \tag{5.32}$$

for all δ sufficiently small. Letting $n \to \infty$ (up to subsequences) in (5.32) yields

$$\left| \int_t^{t+\delta} \int_{\mathbb{R}^d} \psi_\varepsilon(x,\tau)\varphi(x)\, \rho(x)\mathrm{d}x\mathrm{d}\tau - \delta \int_{\mathbb{R}^d} \varphi(x)\, \mathrm{d}\nu(t) \right| \le \frac{\delta^2}{2} \underline{C}\, \|\psi\|_{1,\rho} \left\| \rho^{-1}(-\Delta)^s(\varphi) \right\|_\infty. \tag{5.33}$$

Dividing (5.33) by δ and letting $\delta \to 0$ one deduces that (recall that t is a Lebesgue point for $\psi_\varepsilon(\cdot,t)$)

$$\int_{\mathbb{R}^d} \psi_\varepsilon(x,t)\varphi(x)\, \rho(x)\mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x)\, \mathrm{d}\nu(t)\,,$$

which is valid for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$, whence $\psi_\varepsilon(x,t)\rho(x)\mathrm{d}x = \mathrm{d}\nu(t)$.

We now prove the claimed properties of ψ_ε. Letting $n \to \infty$ in (5.24) (with $t^* = T$ and $t_* = t$) and using the just proved convergence of $\{\rho(\cdot)\psi_{n,\varepsilon}(\cdot,t)\}$ to $\rho(\cdot)\psi_\varepsilon(\cdot,t)$ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$, we get

$$\begin{aligned}&\left| \int_{\mathbb{R}^d} \psi(x)\xi_R(x)\, \rho(x)\mathrm{d}x - \int_{\mathbb{R}^d} \psi_\varepsilon(x,t)\xi_R(x)\, \rho(x)\mathrm{d}x \right| \\ \le& \underline{C}\,(T-t)\, \|\psi\|_{1,\rho}\, c^{-1} \left(R^{-2s} + R^{-2s+\gamma} \right) \|(1+|x|^\gamma)(-\Delta)^s(\xi)\|_\infty\end{aligned} \tag{5.34}$$

for a.e. $t \in (0,T)$, c being as in (1.2). Letting $R \to \infty$ in (5.34) we deduce (5.29). Thanks to (5.21) and (5.29) we infer in particular that

$$\lim_{n\to\infty} \|\psi_{n,\varepsilon}(t)\|_{1,\rho} = \|\psi_\varepsilon(t)\|_{1,\rho} \,,$$

so that the convergence of $\{\rho(\cdot)\psi_{n,\varepsilon}(\cdot,t)\}$ to $\rho(\cdot)\psi_\varepsilon(\cdot,t)$ also takes place in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$. Recalling that $g(\cdot,t)$ belongs to $C_b(\mathbb{R}^d)$ (Lemma A.6), we can let $n \to \infty$ in (5.16) to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} g(x,T)\psi(x)\,\rho(x)\mathrm{d}x - \int_{\mathbb{R}^d} g(x,t)\psi_\varepsilon(x,t)\,\rho(x)\mathrm{d}x \\
&= \lim_{n\to\infty} \left(\int_t^T \int_{\mathbb{R}^d} (a_n(x,\tau)+\varepsilon - a(x,\tau))\,(-\Delta)^s(g)(x,\tau)\,\psi_{n,\varepsilon}(x,\tau)\,\mathrm{d}x\mathrm{d}\tau \right) \\
&= \lim_{n\to\infty} \left(\int_t^T \int_{\mathbb{R}^d} (a_n(x,\tau)+\varepsilon - a(x,\tau))\,(u_2(x,\tau+h)-u_1(x,\tau))\,\psi_{n,\varepsilon}(x,\tau)\,\rho(x)\mathrm{d}x\mathrm{d}\tau \right) \\
&= \varepsilon \int_t^T \int_{\mathbb{R}^d} (u_2(x,\tau+h)-u_1(x,\tau))\,\psi_\varepsilon(x,\tau)\,\rho(x)\mathrm{d}x\mathrm{d}\tau \quad \text{for a.e. } t\in(0,T)\,,
\end{aligned} \tag{5.35}$$

where in the last integral we can pass to the limit since $\{\psi_{n,\varepsilon}\}$ tends to ψ_ε weakly in $L^2_\rho(\mathbb{R}^d\times(t,T))$, $\{a_n\}$ tends to a pointwise with $\{\|a_n\|_{L^\infty(\mathbb{R}^d\times(t,T))}\}$ bounded, and u_1, u_2 belong to $L^p_\rho(\mathbb{R}^d\times(t,T+h))$ for all $p\in[1,\infty]$. In particular, from (5.35) and (5.29) we get (5.31). Notice that, in a similarly way, we can pass to the limit in (5.22) (which actually holds for any $t_*, t^* \in (0,T)$) and get (5.30). ☐

5.4. **Passing to the limit as $\varepsilon \to 0$ and proof of Theorem 3.4.** We are now in position to prove Theorem 3.4, using the strategy of [34]: we give some detail for the reader's convenience.

Proof of Theorem 3.4. To begin with, we introduce the Riesz potential $H_\varepsilon(\cdot,t)$ of $\rho(\cdot)\psi_\varepsilon(\cdot,t)$. Since we only know that $\rho(\cdot)\psi_\varepsilon(\cdot,t)\in L^1(\mathbb{R}^d)$, we have no information over the integrability of $H_\varepsilon(\cdot,t)$ other than $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ (by classical results, see e.g. [28, p. 61]). However, exploiting (5.30) and proceeding once again as in the proof of (4.21), we obtain

$$I_{2s}*(\rho\psi) - H_\varepsilon(\cdot,t) = \int_t^T (a(\cdot,\tau)+\varepsilon)\,\psi_\varepsilon(\cdot,\tau)\,\mathrm{d}\tau \ge 0 \quad \text{for a.e. } t\in(0,T)\,,$$

whence, in particular,

$$0 \le H_\varepsilon(x,t_1) \le H_\varepsilon(x,t_2) \le H_\varepsilon(x,T) = I_{2s}*(\rho\psi)\,(x) \tag{5.36}$$

for a.e. $0<t_1\le t_2\le T$ and a.e. $x\in\mathbb{R}^d$. The above inequality shows that $H_\varepsilon(\cdot,t)$ belongs to $L^p(\mathbb{R}^d)$ at least for the same p for which $H_\varepsilon(\cdot,T)$ does, namely for any $p\in(d/(d-2),\infty]$.

Our next goal is to let $\varepsilon\to 0$ (along a fixed sequence whose index for the moment we omit, in order to improve readability). Thanks to the boundedness of $\{\rho(\cdot)\psi_\varepsilon(\cdot,t)\}$ in $L^1(\mathbb{R}^d)$ (trivial consequence of (5.29)), for a.e. $t\in(0,T)$ there exists a subsequence $\{\varepsilon_n\}$ (a priori depending on t) such that $\{\rho(\cdot)\psi_\varepsilon(\cdot,t)\}$ converges to a positive finite Radon measure $\nu(t)$ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$. In order to overcome the possible dependence of $\{\varepsilon_n\}$ on t, we exploit the properties of $\{H_\varepsilon\}$. First notice that (5.36) ensures the uniform boundedness of $\{H_\varepsilon\}$ in $L^p(\mathbb{R}^d\times(0,T))$ for any $p\in(d/(d-2),\infty]$. This entails the existence of a decreasing subsequence $\{\varepsilon_m\}$ such that $\{H_{\varepsilon_m}\}$ converges weakly in $L^p(\mathbb{R}^d\times(0,T))$ to a suitable limit H. Mazur's Lemma implies that there exists a sequence $\{H_k\}$ of convex combinations of $\{H_{\varepsilon_m}\}$ that converges *strongly* to H in $L^p(\mathbb{R}^d\times(0,T))$. By definition, the sequence $\{H_k\}$ is of the form

$$H_k = \sum_{m=1}^{M_k} \lambda_{m,k} H_{\varepsilon_m}\,, \quad \sum_{m=1}^{M_k} \lambda_{m,k} = 1$$

for some sequence $\{M_k\} \subset \mathbb{N}$ and a suitable choice of the coefficients $\lambda_{m,k} \in [0,1]$. With no loss of generality we shall also assume that

$$\lim_{k\to\infty}\left(\sum_{m=1}^{M_k} \varepsilon_m \lambda_{m,k}\right) = 0\,.$$

This can be justified by applying iteratively Mazur's Lemma on suitable subsequences of $\{H_{\varepsilon_m}\}$. Now notice that the function whose Riesz potential is H_k is

$$f_k(x,t) = \sum_{m=1}^{M_k} \lambda_{m,k}\, \rho(x)\psi_{\varepsilon_m}(x,t)\,.$$

Multiplying (5.31) (with $\varepsilon = \varepsilon_m$) by $\lambda_{m,k}$ and summing over k, one gets that f_k satisfies

$$\begin{aligned} &\left|\int_{\mathbb{R}^d} g(x,T)\psi(x)\,\rho(x)\mathrm{d}x - \int_{\mathbb{R}^d} g(x,t) f_k(x,t)\,\mathrm{d}x\right| \\ &\le \left(\sum_{m=1}^{M_k} \varepsilon_m\lambda_{m,k}\right)(T-t)\,\|\psi\|_{1,\rho}\,\|u_2(\tau+h)-u_1(\tau)\|_{L^\infty(\mathbb{R}^d\times(t,T))} \end{aligned} \tag{5.37}$$

for a.e. $t \in (0,T)$, whereas from (5.29) and (5.34) we infer that

$$\begin{aligned} &\left|\int_{\mathbb{R}^d} \psi(x)\xi_R(x)\,\rho(x)\mathrm{d}x - \int_{\mathbb{R}^d} f_k(x,t)\,\xi_R(x)\,\mathrm{d}x\right| \\ \le& \underline{C}\,(T-t)\,\|\psi\|_{1,\rho}\,c^{-1}\left(R^{-2s} + R^{-2s+\gamma}\right)\|(1+|x|^\gamma)(-\Delta)^s(\xi)\|_\infty \end{aligned} \tag{5.38}$$

for a.e. $t \in (0,T)$ and

$$\int_{\mathbb{R}^d} \psi(x)\,\rho(x)\mathrm{d}x = \int_{\mathbb{R}^d} f_k(x,t)\,\mathrm{d}x \quad \text{for a.e. } t\in(0,T)\,. \tag{5.39}$$

Letting $k \to \infty$ we find that, for a.e. $t \in (0,T)$, there exists a subsequence of $\{f_k(\cdot,t)\}$ (a priori depending on t) that converges in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ to a positive, finite Radon measure $\nu(t)$. But the fact that $\{H_k\}$ converges strongly in $L^p(\mathbb{R}^d \times (0,T))$ to H forces the potential of $\nu(t)$ to coincide a.e. with $H(\cdot,t)$. This is a consequence of [28, Theorem 3.8]. By [28, Theorem 1.12] we therefore deduce that the limit $\nu(t)$ is uniquely determined by its potential $H(\cdot,t)$. This identification allows to assert that for a.e. $t \in (0,T)$ the *whole* sequence $\{f_k(\cdot,t)\}$ converges to $\nu(t)$ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$.

Passing to the limit in (5.36) (after having set $\varepsilon = \varepsilon_m$, multiplied by $\lambda_{m,k}$ and summed over k) we deduce that also the potentials $H(\cdot,t)$ of $\nu(t)$ are ordered and bounded above by $I_{2s} * (\rho\psi)$:

$$0 \le H(x,t_1) \le H(x,t_2) \le I_{2s} * (\rho\psi)\,(x) \quad \text{for a.e. } 0 < t_1 \le t_2 \le T\,,\ \text{for a.e. } x \in \mathbb{R}^d. \tag{5.40}$$

Letting $k \to \infty$ in (5.38) yields

$$\begin{aligned} &\left|\int_{\mathbb{R}^d} \psi(x)\xi_R(x)\,\rho(x)\mathrm{d}x - \int_{\mathbb{R}^d} \xi_R(x)\,\mathrm{d}\nu(t)\right| \\ \le& \underline{C}\,(T-t)\,\|\psi\|_{1,\rho}\,c^{-1}\left(R^{-2s} + R^{-2s+\gamma}\right)\|(1+|x|^\gamma)(-\Delta)^s(\xi)\|_\infty \end{aligned} \tag{5.41}$$

for a.e. $t \in (0,T)$, whence, letting $R \to \infty$ in (5.41), we obtain

$$\int_{\mathbb{R}^d} \psi(x)\,\rho(x)\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{d}\nu(t) \quad \text{for a.e. } t\in(0,T)\,. \tag{5.42}$$

Gathering (5.39) and (5.42) we infer that $\{f_k(\cdot,t)\}$ converges to $\nu(t)$ also in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$: this allows us to pass to the limit in (5.37) to get (by exploiting (5.4) as well) identity (5.8). As a consequence of the monotonicity given by (5.40) and thanks to (5.41)-(5.42), the curve $\nu(t)$ can be extended to *every* $t \in (0,T]$ so that it still satisfies (5.40)-(5.42) (one uses again [28, Theorem 3.8]). Recalling that $g(x,t) = U_2(x,t+h) - U_1(x,t)$ and that potentials do not increase in time (Lemma

5.1), we have that $g(x,t) \le U_2(x,h) - U_1(x,t_0)$ holds for all $x \in \mathbb{R}^d$ and all $t_0 > t$. Because $\nu(t)$ is a positive finite Radon measure, this fact and (5.8) imply that

$$\int_{\mathbb{R}^d} g(x,T)\psi(x)\,\rho(x)\mathrm{d}x \le \int_{\mathbb{R}^d} (U_2(x,h) - U_1(x,t_0))\,\mathrm{d}\nu(t) \quad \forall t_0 > t\,. \tag{5.43}$$

Our next goal is to let t tend to zero in (5.43). Since the mass of $\nu(t)$ is constant (formula (5.42)), up to subsequences $\nu(t)$ converges to a suitable positive finite Radon measure ν in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ as $t \downarrow 0$. Moreover, by (5.40), we know that the potentials $H(\cdot,t)$ of $\nu(t)$ are nondecreasing in t (for a.e. x): in particular, $H(\cdot,t)$ admits a pointwise limit almost everywhere H_0 as $t \downarrow 0$. Theorem 3.8 of [28] ensures that H_0 coincides almost everywhere with the potential of the limit measure ν (which therefore does not depend on the subsequence). We can then pass to the limit in the integral

$$\int_{\mathbb{R}^d} U_1(x,t_0)\,\mathrm{d}\nu(t)\,. \tag{5.44}$$

Indeed, by Fubini's Theorem, (5.44) is equal to

$$\int_{\mathbb{R}^d} u_1(x,t_0)H(x,t)\,\rho(x)\mathrm{d}x\,. \tag{5.45}$$

Passing to the limit in (5.45) as $t \downarrow 0$ we get that

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^d} u_1(x,t_0)H(x,t)\,\rho(x)\mathrm{d}x = \int_{\mathbb{R}^d} u_1(x,t_0)H_0(x)\,\rho(x)\mathrm{d}x \tag{5.46}$$

by dominated convergence. Recalling that H_0 is the potential of ν, and using again Fubini's Theorem, (5.46) can be rewritten as

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^d} U_1(x,t_0)\,\mathrm{d}\nu(t) = \int_{\mathbb{R}^d} U_1(x,t_0)\,\mathrm{d}\nu\,.$$

One proceeds similarly for the integral

$$\int_{\mathbb{R}^d} U_2(x,h)\,\mathrm{d}\nu(t)\,.$$

Hence, passing to the limit as $t \downarrow 0$ in (5.43) yields

$$\int_{\mathbb{R}^d} g(x,T)\psi(x)\,\rho(x)\mathrm{d}x \le \int_{\mathbb{R}^d} (U_2(x,h) - U_1(x,t_0))\,\mathrm{d}\nu \quad \forall t_0 > 0\,. \tag{5.47}$$

Now we let $t_0 \downarrow 0$ in (5.47). By monotone convergence (Lemmas 5.1 and 5.2) we obtain

$$\int_{\mathbb{R}^d} g(x,T)\psi(x)\,\rho(x)\mathrm{d}x \le \int_{\mathbb{R}^d} (U_2(x,h) - U^\mu(x))\,\mathrm{d}\nu\,; \tag{5.48}$$

in this step it is crucial that the limit of $U_1(x,t_0)$ to $U^\mu(x)$ is taken *for every* $x \in \mathbb{R}^d$ (Lemma 5.2), because we have no information over ν besides the fact that it is a positive finite Radon measure. Still by monotonicity we have that $U_2(x,h) \le U^\mu(x)$ for every $x \in \mathbb{R}^d$. Thus, from (5.48) it follows that

$$\int_{\mathbb{R}^d} g(x,T)\psi(x)\,\rho(x)\mathrm{d}x \le 0\,. \tag{5.49}$$

Since (5.49) holds for any $h, T > 0$ and any $\psi \in \mathcal{D}_+(\mathbb{R}^d)$, we infer that $U_2 \le U_1$. By interchanging the role of u_1 and u_2 we get that $U_1 \le U_2$, whence $U_1 = U_2$ and $u_1 = u_2$. □

Appendix A.

We recall here some basic properties of the fractional Laplacian (and of a similar nonlocal, nonlinear operator) of functions in $\mathcal{D}(\mathbb{R}^d)$. We omit the proofs of the first two lemmas, since they follow by exploiting the same strategy of [5, Lemma 2.1].

Lemma A.1. *The s-Laplacian $(-\Delta)^s(\phi)(x)$ of any $\phi \in \mathcal{D}(\mathbb{R}^d)$ is a regular function which decays (together with its derivatives) at least like $|x|^{-d-2s}$ as $|x| \to \infty$.*

Lemma A.2. *For any $\phi \in \mathcal{D}(\mathbb{R}^d)$, the function*

$$l_s(\phi)(x) := \int_{\mathbb{R}^d} \frac{(\phi(x) - \phi(y))^2}{|x-y|^{d+2s}} \, \mathrm{d}y \quad \forall x \in \mathbb{R}^d$$

is regular and decays (together with its derivatives) at least like $|x|^{-d-2s}$ as $|x| \to \infty$.

Lemma A.3. *For any $R > 0$, let ξ_R be the cut-off function*

$$\xi_R(x) := \xi\left(\frac{x}{R}\right) \quad \forall x \in \mathbb{R}^d \, ,$$

where $\xi(x)$ is a positive, regular function such that $\|\xi\|_\infty \le 1$, $\xi \equiv 1$ in B_1 and $\xi \equiv 0$ in B_2^c. Then, $(-\Delta)^s(\xi_R)$ and $l_s(\xi_R)$ enjoy the following property:

$$(-\Delta)^s(\xi_R)(x) = \frac{1}{R^{2s}}(-\Delta)^s(\xi)\left(\frac{x}{R}\right), \quad l_s(\xi_R)(x) = \frac{1}{R^{2s}} l_s(\xi)\left(\frac{x}{R}\right) \quad \forall x \in \mathbb{R}^d \, .$$

Proof. We only prove the result for $l_s(\xi_R)$, since the proof for $(-\Delta)^s(\xi_R)$ is identical. Letting $\widetilde{y} = y/R$, one has:

$$l_s(\xi_R)(x) = \int_{\mathbb{R}^d} \frac{(\xi_R(x) - \xi_R(y))^2}{|x-y|^{d+2s}} \, \mathrm{d}y = \frac{1}{R^{2s}} \int_{\mathbb{R}^d} \frac{(\xi(x/R) - \xi(\widetilde{y}))^2}{|x/R - \widetilde{y}|^{d+2s}} \, \mathrm{d}\widetilde{y} = \frac{1}{R^{2s}} l_s(\xi)\left(\frac{x}{R}\right).$$

□

The next lemmas contain technical ingredients concerning fractional Sobolev spaces and Riesz potentials, which we need in the proofs of our existence and uniqueness results.

Lemma A.4. *Let $d > 2s$ and assume that ρ satisfies (1.2) for some $\gamma \in (0, d+2s]$. Consider a function $v \in L^2_{\mathrm{loc}}((0,\infty); \dot{H}^s(\mathbb{R}^d))$ such that, for all $t_2 > t_1 > 0$,*

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |v(x,t)|^2 \, \rho(x) \mathrm{d}x \mathrm{d}t \le C_0 \, , \tag{A.1}$$

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left|(-\Delta)^{\frac{s}{2}} (v) \, (x,t)\right|^2 \mathrm{d}x \mathrm{d}t \le C_0 \tag{A.2}$$

and

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |v_t(x,t)|^2 \, \rho(x) \mathrm{d}x \mathrm{d}t \le C_0 \, , \tag{A.3}$$

where C_0 is a positive constant depending only on t_1 and t_2. Take any cut-off functions $\xi_1 \in C_c^\infty(\mathbb{R}^d)$, $\xi_2 \in C_c^\infty((0,\infty))$ and define $v_c : \mathbb{R}^d \to \mathbb{R}$ as follows:

$$v_c(x,t) := \xi_1(x) \xi_2(t) v(x,t) \quad \forall (x,t) \in \mathbb{R}^d \times \mathbb{R} \, ,$$

where we implicitly assume ξ_2 and v to be zero for $t < 0$. Then

$$\|v_c\|^2_{H^s(\mathbb{R}^{d+1})} = \|v_c\|^2_{L^2(\mathbb{R}^{d+1})} + \|v_c\|^2_{\dot{H}^s(\mathbb{R}^{d+1})} \le C' \tag{A.4}$$

for a positive constant C' that depends only on ρ, ξ_1 and ξ_2 (also through C_0).

Proof. The validity of

$$\|v_c\|^2_{L^2(\mathbb{R}^{d+1})} \le C' \tag{A.5}$$

is an immediate consequence of (A.1) and of the fact that ρ is bounded away from zero on compact sets (from now on C' will be a constant as in the statement that may change from line to line). Moreover, since $(v_c)_t = \xi_1\xi_2' v + \xi_1\xi_2 v_t$, by (A.1), (A.3) and again the fact that ρ is bounded away from zero on compact sets we deduce that

$$\|(v_c)_t\|^2_{L^2(\mathbb{R}^{d+1})} \le C' \, . \tag{A.6}$$

Now we have to handle the spatial regularity of v_c. Straightforward computations show that

$$\begin{aligned} \|v_c(t)\|^2_{\dot H^s(\mathbb{R}^d)} = & \frac{C_{d,s}}{2}\,\xi_2^2(t)\int_{\mathbb{R}^d}\xi_1^2(x)\left(\int_{\mathbb{R}^d}\frac{(v(x,t)-v(y,t))^2}{|x-y|^{d+2s}}\,\mathrm{d}y\right)\mathrm{d}x \\ & + \frac{C_{d,s}}{2}\,\xi_2^2(t)\int_{\mathbb{R}^d}|v(y,t)|^2\left(\int_{\mathbb{R}^d}\frac{(\xi_1(x)-\xi_1(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}x\right)\mathrm{d}y \\ & + C_{d,s}\,\xi_2^2(t)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\xi_1(x)v(y,t)\frac{(v(x,t)-v(y,t))\,(\xi_1(x)-\xi_1(y))}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y \, . \end{aligned} \tag{A.7}$$

The Cauchy-Schwarz inequality allows us to bound the third integral on the r.h.s. of (A.7) by the first two integrals. As concerns the first one, we have:

$$\frac{C_{d,s}}{2}\,\xi_2^2(t)\int_{\mathbb{R}^d}\xi_1^2(x)\left(\int_{\mathbb{R}^d}\frac{(v(x,t)-v(y,t))^2}{|x-y|^{d+2s}}\,\mathrm{d}y\right)\mathrm{d}x \le \chi_{\operatorname{supp}\xi_2}(t)\,\|\xi_2\|^2_\infty\,\|\xi_1\|^2_\infty\,\|v(t)\|^2_{\dot H^s(\mathbb{R}^d)} \, . \tag{A.8}$$

In order to bound the second integral, it is important to recall that the function $l_s(\xi_1)(y)$ is regular and decays at least like $|y|^{-d-2s}$ as $|y|\to\infty$ (for the definition and properties of l_s see Lemmas A.2 and A.3). Hence, thanks to the assumptions on ρ and γ, we infer that

$$\xi_2^2(t)\int_{\mathbb{R}^d}|v(y,t)|^2\left(\int_{\mathbb{R}^d}\frac{(\xi_1(x)-\xi_1(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}x\right)\mathrm{d}y \le C'\chi_{\operatorname{supp}\xi_2}(t)\,\|\xi_2\|^2_\infty\int_{\mathbb{R}^d}|v(y,t)|^2\,\rho(y)\mathrm{d}y \, . \tag{A.9}$$

Integrating in time (A.7), using (A.8), (A.9), (A.1), (A.2) and recalling the validity of the identity $\left\|(-\Delta)^{\frac{s}{2}}(v_c)(t)\right\|^2_{L^2(\mathbb{R}^d)} = \|v_c(t)\|^2_{\dot H^s(\mathbb{R}^d)}$, we then get

$$\left\|(-\Delta)^{\frac{s}{2}}(v_c)\right\|^2_{L^2(\mathbb{R}^{d+1})} \le C' \, . \tag{A.10}$$

By exploiting (A.5), (A.6) and (A.10) one deduces (A.4), e.g. by using Fourier transform methods. □

Lemma A.5. *Let $d>2s$ and $\phi:\mathbb{R}^d\to\mathbb{R}$ be a continuous function which belongs to $L^1(\mathbb{R}^d)$ and decays at least like $|x|^{-d}$ as $|x|\to\infty$. Then, the convolution $I_{2s}*\phi$ (namely, the Riesz potential of ϕ) is also a continuous function, decaying at least like $|x|^{-d+2s}$ as $|x|\to\infty$.*

Proof. The idea of the proof is to split the convolution $(I_{2s}*\phi)(x)$ in the three regions $B^c_{2|x|}(0)$, $B_{|x|/2}(x)$, $B_{2|x|}(0)\setminus B_{|x|/2}(x)$ and use there the decay and integrability properties of ϕ and I_{2s}. We omit the details. □

Lemma A.6. *Let $d>2s$ and assume that ρ satisfies* (1.2) *for some $\gamma\in(0,2s)$. Let $v\in L^1_\rho(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ and U^v_ρ be the Riesz potential of ρv. Then U^v_ρ belongs to $C(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$ for all p such that*

$$p\in\left(\frac{d}{d-2s},\infty\right] . \tag{A.11}$$

Proof. In order to prove that U_ρ^v belongs to $C(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$ for all p satisfying (A.11), we proceed as follows:

$$U_\rho^v(x)=\underbrace{\int_{B_1(0)}\rho(y)\,v(y)\,I_{2s}(x-y)\,\mathrm{d}y}_{U_{\rho,1}^v(x)}+\underbrace{\int_{\mathbb{R}^d}\chi_{B_1^c(0)}(y)\,\rho(y)\,v(y)\,I_{2s}(x-y)\,\mathrm{d}y}_{U_{\rho,2}^v(x)}\,.$$

Exploiting the fact that $v\in L^\infty(\mathbb{R}^d)$ and $\gamma<2s$ (so that $|y|^{-d+2s}\rho(y)$ is locally integrable), it is easily seen that $U_{\rho,1}^v(x)$ is a continuous function which decays at least like $|x|^{-d+2s}$ as $|x|\to\infty$. In particular, it belongs to $L^p(\mathbb{R}^d)$ for all p satisfying (A.11). As concerns $U_{\rho,2}^v(x)$, notice that since $v\in L^1_\rho(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ we have that the function $\chi_{B_1^c(0)}\rho v$ belongs to $L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Hence $U_{\rho,2}^v(x)$ is continuous too. To prove that it belongs to $L^p(\mathbb{R}^d)$ for all p satisfying (A.11), we write:

$$U_{\rho,2}^v=\left(\chi_{B_1(0)}\,I_{2s}\right)*\left(\chi_{B_1^c(0)}\rho v\right)+\left(\chi_{B_1^c(0)}\,I_{2s}\right)*\left(\chi_{B_1^c(0)}\rho v\right);\tag{A.12}$$

since $\chi_{B_1(0)}\,I_{2s}\in L^1(\mathbb{R}^d)$ and $\chi_{B_1^c(0)}\rho v\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, the first convolution in (A.12) belongs to $L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Using the fact that $\chi_{B_1^c(0)}\,I_{2s}\in L^p(\mathbb{R}^d)$ for all p as in (A.11) and $\chi_{B_1^c(0)}\rho v\in L^1(\mathbb{R}^d)$, we infer that the second convolution in (A.12) belongs to $L^p(\mathbb{R}^d)$ for all such p. The latter property is then inherited by $U_{\rho,2}^v$. □

Appendix B.

This section is devoted to give a sketch of the proofs of Theorem 3.7 and of the forthcoming Proposition B.1.

Sketch of proof of Theorem 3.7. We start from the validity of the fractional "integration by parts" formula

$$\frac{C_{d,s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(\phi(x)-\phi(y))(\psi(x)-\psi(y))}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y=\int_{\mathbb{R}^d}\phi(x)(-\Delta)^s(\psi)(x)\,\mathrm{d}x\tag{B.1}$$

for all $\phi,\psi\in\mathcal{D}(\mathbb{R}^d)$, and our aim is to extend it to all functions of $X_{s,\rho}$. In order to do it, the first step consists in showing that $C^\infty(\mathbb{R}^d)\cap X_{s,\rho}$ is dense in $X_{s,\rho}$. This can be done by mollification arguments, which however are slightly more complicated than the standard ones, since we work with the weighted spaces $L^2_\rho(\mathbb{R}^d)$ and $L^2_{\rho^{-1}}(\mathbb{R}^d)$ instead of $L^2(\mathbb{R}^d)$. Hence, given $v,w\in C^\infty(\mathbb{R}^d)\cap X_{s,\rho}$, one plugs the cut-off functions $\phi:=\xi_R v$ and $\psi:=\xi_R w$ into (B.1) and lets $R\to\infty$. The problem is that on the r.h.s. there appear terms involving $\|\xi_R w\|_{\dot H^s}$, and a priori we do not know whether $C^\infty(\mathbb{R}^d)\cap X_{s,\rho}$ is continuously embedded in $\dot H^s(\mathbb{R}^d)$. But this turns out to be true: the inequality

$$\frac{C_{d,s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(w(x)-w(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y\le\int_{\mathbb{R}^d}w(x)(-\Delta)^s(w)(x)\,\mathrm{d}x\quad\forall w\in C^\infty(\mathbb{R}^d)\cap X_{s,\rho}\tag{B.2}$$

can be proved just by repeating the above scheme with $\phi=\psi=\xi_R w$. In fact, on the r.h.s. of (B.1) we still have terms involving $\|\xi_R w\|_{\dot H^s}$, but the latter are small and can be absorbed into the l.h.s.; passing to the limit as $R\to\infty$ yields (B.2). Therefore, we can now let $R\to\infty$ safely in (B.1) (with $\phi=\xi_R v$ and $\psi=\xi_R w$) and obtain that

$$\frac{C_{d,s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y=\int_{\mathbb{R}^d}v(x)(-\Delta)^s(w)(x)\,\mathrm{d}x\tag{B.3}$$

for all $v,w\in C^\infty(\mathbb{R}^d)\cap X_{s,\rho}$, which in particular shows that (B.2) is actually an equality. Notice that in all these approximation procedures using cut-off functions, to prove that "remainder" terms go to zero we deeply exploit the results provided by Lemmas A.1, A.2 and A.3. It is in fact here that the condition $\gamma<2s$ plays a fundamental role: in particular, it ensures that both $\|\rho^{-1}(-\Delta)^s(\xi_R)\|_\infty$ and $\|\rho^{-1}l_s(\xi_R)\|_\infty$ vanish as $R\to\infty$. As already mentioned, we refer the reader to the note [31] for the details. However, for similar computations involving $(-\Delta)^s(\xi_R)$ and $l_s(\xi_R)$, see also the proofs of Proposition 4.1, Lemma 4.3 and Lemma 5.5.

By the claimed density of $C^\infty(\mathbb{R}^d) \cap X_{s,\rho}$, we are allowed to extend (B.3) to the whole of $X_{s,\rho}$. Clearly, the r.h.s. of (B.3) can be rewritten as

$$\int_{\mathbb{R}^d} v(x)\, A(w)(x)\, \rho(x) \mathrm{d}x \,,$$

and letting $v = w$ we obtain that the operator A is positive. The fact that it is densely defined is trivial since, for instance, $\mathcal{D}(\mathbb{R}^d) \subset X_{s,\rho}$. Because in (B.3) one can interchange the role of v and w, we also have that A is symmetric. In order to prove that it is self-adjoint we need to show that $D(A^*) \subset D(A)$, namely that any function of $D(A^*)$ also belongs to $X_{s,\rho}$. It is indeed straightforward to check this fact, and we leave it to the reader.

We finally deal with the quadratic form Q associated to A. Thanks to (B.3), we have that

$$Q(v,v) = \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(x)-v(y))^2}{|x-y|^{d+2s}} \, \mathrm{d}x \mathrm{d}y \quad \forall v \in D(A) \,. \tag{B.4}$$

As it is well known (see e.g. [13]), the domain $D(Q)$ of Q is just the closure of $D(A)$ w.r.t. the norm

$$\|v\|_Q^2 := \|v\|_{2,\rho^{-1}}^2 + Q(v,v) = \|v\|_{2,\rho^{-1}}^2 + \|v\|_{\dot{H}^s}^2 \,.$$

It is then easy to see that such a closure is nothing but $L^2_\rho(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$ and the quadratic form on $D(Q) = L^2_\rho(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$ is still represented by (B.4).

By classical results (we refer again to [13]), proving that A generates a Markov semigroup is equivalent to proving that if v belongs to $D(Q)$ then both $v \vee 0$ and $v \wedge 1$ belong to $D(Q)$ and satisfy

$$Q(v \vee 0, v \vee 0) \le Q(v,v) \,, \quad Q(v \wedge 1, v \wedge 1) \le Q(v,v) \,.$$

But the latter properties are straightforward consequences of the characterization of Q given above.

The last assertions follow from the general theory of symmetric Markov semigroups (cf. [13, Section 1.4]) and from their known analiticity properties (cf. [13, Theorem 1.4.2]). See also the discussion in the proof of Lemma 5.3. □

The next proposition extends the symmetry property of the operator $A = \rho^{-1} (-\Delta)^s$ to functions which belong to other suitable L^p_ρ spaces. This is essential in proving our uniqueness Theorem 3.4 for certain values of γ and s in low dimensions $d \le 3$, more precisely whenever $(d-\gamma)/(d-2s) > 2$.

Proposition B.1. *Let $d > 2s$ and assume that ρ satisfies (1.2) for some $\gamma \in [0,2s) \cap [0, d-2s]$ and $\gamma_0 \in [0,\gamma]$. Let $p \in [2, 2(d-\gamma)/(d-2s))$ and $p' = p/(p-1)$ be its conjugate exponent. Suppose that $v, w \in L^p_\rho(\mathbb{R}^d)$ are such that $A(v), A(w) \in L^{p'}_\rho(\mathbb{R}^d)$. Then $v, w \in \dot{H}^s(\mathbb{R}^d)$ and the following formula holds:*

$$\begin{aligned} \int_{\mathbb{R}^d} v(x)(-\Delta)^s(w)(x)\, \mathrm{d}x &= \int_{\mathbb{R}^d} (-\Delta)^s(v)(x)\, w(x)\, \mathrm{d}x \\ &= \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{d+2s}} \, \mathrm{d}x \mathrm{d}y \,. \end{aligned}$$

Sketch of proof. The method of proof proceeds along the lines of the one of Theorem 3.7. The main difference here lies in the fact that, when using the approximation procedure by cut-off functions mentioned above, if p is *strictly larger* than 2 in order to prove that "remainder" terms go to zero one cannot exploit the fact that $\rho^{-1}(-\Delta)^s(\xi_R)$ and $\rho^{-1} l_s(\xi_R)$ vanish in $L^\infty(\mathbb{R}^d)$ as $R \to \infty$. In fact, such remainder terms are of the form

$$\int_{\mathbb{R}^d} v^2(x)(-\Delta)^s(\xi_R)(x)\, \mathrm{d}x \quad \text{or} \quad \int_{\mathbb{R}^d} v^2(x)\, l_s(\xi_R)(x)\, \mathrm{d}x \,. \tag{B.5}$$

Thanks to Lemmas A.1, A.2 and A.3, it is direct to see that $\|\rho^{-1}(-\Delta)^s(\xi_R)\|_{q,-\gamma}$ and $\|\rho^{-1} l_s(\xi_R)\|_{q,-\gamma}$ vanish as $R \to \infty$ provided $q > (d-\gamma)/(2s-\gamma)$, whence the condition $p \in [2, 2(d-\gamma)/(d-2s))$ to ensure that also the integrals in (B.5) go to zero as $R \to \infty$. □

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