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Report on the dissertation of Karol Duda

Dynamics and computability in Geometric Group Theory

The dissertation consists of two parts:

- Part I. Computable aspects of amenability
- Part II. Locally elliptic actions on small cancellation complexes

Both themes (amenability and actions of groups on complexes) belong to actively developing areas of geometric group theory with many interesting results and open problems. We characterize the main results of this dissertation.

Part I contains 5 results (see below). Results 1-4 are obtained in the joint paper with A. Ivanov, which is appended to the dissertation. They are formulated in Chapters 2 and 3 without repeating the proofs. Result 5 is formulated and proved in Chapter 4.

Result 1 (Theorem I.1 below). A well known Halls' marriage Theorem from combinatorics and graph theory gives necessary and sufficient conditions for the existence of a left perfect $(1,1)$-matching in a finite bipartite graph. It has a generalization for the case of locally finite bipartite graphs and perfect $(1,d)$-matchings there d is any natural number. This generalization is called Hall's harem theorem and can be found in the book of Ceccerini-Silberstein, Grigorchuk and Pierre de la Harpe "Amenability and paradoxical decompositions for pseudogroups and discrete metric spaces".

Passing to a computable version of the latter theorem, the author first recalls definitions of a computable and a highly computable graph, and of a computable $(1,d)$-matching. The first two definitions were introduced by Kierstead, and the third one is the author's definition that naturally generalizes Kierstead's definition of a computable $(1,1)$-matchings. Kierstead proved a computable version of Hall's marriage theorem for locally finite graph and the author (joint paper with A. Ivanov) proved a computable version of Hall's harem theorem. In a simplified form

Theorem I.1 says that if $\Gamma = (\mathbb{N}, \mathbb{N}, E)$ is a *highly computable graph*, satisfying the condition of Hall' harem theorem with the factor $(d+1)$, then Γ has a *computable* perfect $(1,d)$-matching.

Results 2-4 (Theorems I.2, I.3 and I.4 below). The author (joint with A. Ivanov) uses the computable version of Hall's harem theorem (see above) to prove two computable versions of Tarski's alternative theorem about amenability. Recall that amenability is equivalent to Fϕlner's condition. Thus, Tarski's alternative theorem can be formulated as follows: An action of a group G on a set X by permutations does not satisfy Fϕlner's condition if and only if X has a paradoxical G-decomposition.

Theorem I.2 says that if G is a pseudogroup of *computable* transformations of a countable set X which does not satisfy Fϕlner's condition, then X has an *effective* paradoxical G-decomposition.

Theorem I.3 says that if G is a group of *computable* permutations of a countable set X which does not satisfy Fϕlner's condition, then X has an *effective* paradoxical G-decomposition.

Note that these two theorems have very similar proofs, though they cannot be formally deduced one from the other. Indeed, groups and pseudogroups are different notions, and definitions of paradoxical decompositions for them are also different.

Theorem I.4 is rather technical; it says that the family of finite sets appearing in the extended formulation of Theorem I.3. is computable for any finitely generated group G.

The most attractive and nontrivial result among all results of Part I is Result 5.

Result 5 (Theorem 4.1.3 below). Before we formulate this result, we recall some notions. Suppose that $\Gamma = (\mathbb{N}, \mathbb{N}, E)$ is a bipartite graph. Then any perfect $(1, d)$-matching $\mathcal{E}$ in Γ naturally determines a (d to 1)-function $f : \mathbb{N} \to \mathbb{N}$. We call it the *function associated* with $\mathcal{E}$. Studying dynamical properties of this function is important for constructing new nice computable matchings and for deriving computable variants of classical theorems about amenability. This forces the author to introduce a special property of f called *controlled sizes of cycles*. In particular, if $f : \mathbb{N} \to \mathbb{N}$ has this property, then every natural number is a pre-periodic point of f, and one can estimate the sizes of tails and cycles for every initial number $n \in \mathbb{N}$. Theorem 4.1.3 states that under some natural conditions on the bipartite graph Γ one can find a matching $\mathcal{E}'$ in Γ, whose associated function f has controlled sizes of cycles. More precisely,

Theorem 4.1.3 says that if $\Gamma = (\mathbb{N}, \mathbb{N}, E)$ is a locally finite bipartite graph, which satisfies Hall's d-harem condition and some additional natural conditions (like Γ is fully reflected and does not contain edges of kind (n, n)), then there exists a perfect $(1, d-1)$-matching $\mathcal{E}'$, whose associated function $f : \mathbb{N} \to \mathbb{N}$ has controlled sizes of cycles.

This theorem is interesting for its own sake since it gives an additional information for a bit weaker matching. In Section 4.6, the author explains that this theorem is a pre-step to its computable version, which can be used to obtain a computable version of a very general theorem of Schneider about amenability of coarse spaces. Note that the author posted both computable versions in the ArXive but did not included their proofs in this dissertation.

The proof of Theorem 4.1.3 given in this dissertation is quite complex. I think that the exposition of this proof is not optimal – in some places it is written very formal, so that it takes a lot of time to understand it.

Part II contains many interesting results concerning actions of groups on small cancellation complexes.

Theorem II.1 says that torsion subgroups of groups defined by $C(6)$, $C(4) - T(4)$, and $C(3) - T(6)$ small cancellation presentations are finite cyclic groups.

This theorem immediately follows from the following one.

Theorem II.2. Let X be a simply connected $C(6)$, $C(4) - T(4)$, or $C(3) - T(6)$ small cancellation complex. Let G be a group acting on X by automorphisms such that the action induces a free action on the 1-skeleton X^1 of X. If the action is locally elliptic, then G is a finite cyclic group. In particular, G fixes a 2-cell of X.

Note that the proof of this theorem in the cases $C(6)$, $C(4) - T(4)$ demanded a detailed analysis of geodesics in related complexes and is not easy; the case $C(3) - T(6)$ follows relatively easy from Theorem II.3 and Corollary II.4 below.

Theorem II.3. Let X be a simply connected $C(3)-T(6)$-calcellation complex. Then there exists a metric on X turning it into a $CAT(0)$ triangle complex $\mathfrak{X}$ such that every automorphism of X induces an automorphism of $\mathfrak{X}$.

This theorem was proved by using a result of Pride that implies that any simply connected $T(6)$-complex is a polygonal complex. Putting a reasonable metric on the barycentric subdivision of this complex the author checks that it satisfies the link condition and hence is $CAT(0)$. Though the proof is short, it seems that this useful theorem was never mentioned before.

Corollary II.4. A finitely generated group acting locally elliptically on a simply connected $C(3)-T(6)$ small cancellation complex fixes a point.

This corollary follows from the proof of Theorem II.3 and the paper of Norin, Osajda and Przytycki of 2022. As mentioned above, this corollary was used for the analysis of the $C(3)-T(6)$ case in the proof of Theorem II.2. Moreover, this corollary strengthens this theorem in the case of *finitely generated* $C(3)-T(6)$-groups. Indeed, in this case the assumption about freeness of action of G on the the 1-skeleton of X is not anymore required.

Corollary II.4 is nice, and it would be interesting to study, whether the same is valid in the cases of $C(6)$ and $C(4)-T(4)$ small cancellation complexes.

The second corollary from Theorem II.3 concerns the so called Tits alternative. Recall that a group satisfies the Tits alternative if each of its finitely generated subgroups either contains a free nonabelian subgroup or is virtually solvable. This alternative was established (by other authors) for hyperbolic groups, mapping class groups, linear groups, $Out(F_n)$, and for some other groups.

Corollary II.5. Let G be a group acting almost freely on a simply connected $C(3)-T(6)$ small cancellation complex. Then G is virtually cyclic, or virtually $\mathbb{Z}^2$, or contains a nonabelian free group.

This corollary follows straightforwardly from Theorem II. 3 and a result of Osajda and Prytula of 2018.

The last theorem is about homomorphisms from a locally compact group to another group G. The case where G acts acylindrically and coboundedly on a hyperbolic space was considered by Bogopolski and Corson (see their paper in Mathematische Annalen, 2022). The author considers the case of small cancellation groups. Note that every non virtually cyclic group G that admits a proper isometric action on a proper $CAT(0)$-space with G having at least one rank-1 element is acylindrically hyperbolic. I think that the author should cite this result of Bogopolski and Corson since it is related to this theorem.

Theorem II.6. Let G be a group acting geometrically on a locally finite, simply connected $C(6)$, $C(4)-T(4)$, or $C(3)-T(6)$ small cancellation complex X such that the action induces a free action on the 1-skeleton of X. If H is a subgroup of G then any homomorphism $\varphi : L \to H$ from a locally compact group L is continuous or there exists a normal open subgroup $N \subseteq L$ such that $\varphi(N)$ is a finite group.

In the proof of this theorem the author uses a very strong result of Keppeler, Möller and Varghese (2022). In the cases of $C(6)$, $C(4)-T(4)$ small cancellation complexes the proof is a combination of this result and results of other authors. In the case $C(3)-T(6)$ the proof additionally uses author's Theorem II.2.

The minor remarks listed below are not essential for the validity of the proofs.

Summarising:

- The author obtained many interesting and new results in geometric and computational group theory. Some of them are very stimulating for further research in these areas.
- The proofs utilize a modern mathematical technique and, in some places, they are very tricky that shows a high scientific potential of the author.

Based on this, I recommend this dissertation to be accepted.

Referee: Prof. Dr. Habil. Oleg Bogopolski

Remarks

1. Page 17.

The definition of a path is not precise: it is unclear whether x_i are vertices or edges.

2. Page 17.

According to a classical definition in graph theory, a *matching* in an undirected graph is a subset of edges without common vertices.

So, Definition 2.1.1 generalizing this notion to $(1, k)$-*matchings* seems to be not correctly formulated. One should write there that different $(1, k)$-fans of the matching do not have common vertices. This inaccuracy appears since the author defines a fan as a set of edges.

3. Page 18.

In conditions (1)-(2), the matching M is considered as a subset of E, while in Definition 2.1.1 a matching is a collection of subsets of E. However, this inaccuracy does not effect on understanding.

4. Page 20. In Definition 2.3.6.

1) Missed letter $\mathcal{E}$ after the word collection.

2) One should define Δ_X or say that the diagonal Δ_X belongs to $\mathcal{E}$.

5. Page 20.

It might be useful to note that Definition 2.3.8 is a generalization of Definition 2.2.3.

6. Page 20, line -4.

There must be Δ_X instead of δ_X.

7. Page 21. In the definition of a highly computable graph.

It is missed there that this graph must be locally finite. This is essential in Theorem I.1.

8. Page 21.

The letter V is used in two different contexts in and before Definition 3.1.3. Formally, this is correct, but it can lead to a misunderstanding.

9. Page 22.

The Definition 3.1.3 seems to be quite technical and it takes time to understand why it has this form.

It could be helpful to write that Kierstead's theorem is also valid under a stronger (and less technical) condition like

- *for any finite set* $X \subset U$, *we have* $|N(X)| \geqslant 2|X|$.

The same remark to Definition 3.1.4:

- *for any finite set* $X \subset U$, *we have* $|N(X)| \geqslant (d+1)|X|$,
- *for any finite set* $Y \subset V$, *we have* $|N(Y)| \geqslant \frac{1}{d+1}|Y|$,

10. Page 22.

In Definition 3.1.4, one should cite Definition 2.8 of [DI22a] and not Theorem 2.9 of [DI22a].

11. Page 22.

Page 22, line -6

between subsets S and T **of** X.

12. Page 23.

In Theorem I.2, one should say clearly that $X = \mathbb{N}$ or to write "defined on an infinite countable set X".

13. Page 26, line $+18$

Let $f : \mathbb{N} \to \mathbb{N}$ be a function. If for some $i \neq 0$ and $u \in \mathbb{N}$ we have ...

14. Pages 25 and 26.

There is a confusion with names of theorems in Chapter 4.

The title of Section 4.1.3 includes the words Main theorem. One expects that this the main theorem of Chapter 4. However, there is no such name in the body of this section. I guess that this is Theorem 4.1.3 where in brackets stands Main theorem of [Dud21a]. Meanwhile, Main theorem in [Dud21a] is called Theorem 2.4 there.

15. Page 27, line $+1$

In Theorem 4.1.3 the notion of a matching *realizing* a function is used. One can guess what does it mean by reading the first paragraph of 4.1.3. However this notion must be exactly defined before this theorem.

16. Page 55.

Theorem 5.1.4 is not correctly formulated. A correct formulation is like the following:

..., then there exists a reduced disc diagram $D \to X$ and a map $P \to D$, which is a boundary cycle of D, such that $P \to X$ is the composition $P \to D \to X$.

17. Page 57, line $+25$.

"Since X is $C(4) - T(4)$"

This was not explicitly assumed.

18. Page 57, line $+25$.

... is connected **by Proposition 5.1.8**

19. Page 61. lines $+14$, -3.

It is not said what is Y. I guess that Y is a systolic complex.

20. Page 68, line -15.

In this section the assumption that G is a finitely generated group is not used.

21. Page 70, line +26.

Delete s in $x^D s$.

22. Page 77, line +3.

In Section 6.4, we don't need the assumption that G is finitely generated.

23. Page 77, lines −12, −13.

f^{2k} and f^{3k}

24. Page 79, line +4.

Suppose that there exists another $v' \in \mathrm{Fix}_Y(g)$.

25. Page 79, lines +7 – +9.

This is an easy place, but the explanation is not clear.

Maybe so: The initial and the terminal vertices of the path α are the same. Therefore this path is not a geodesic that contradicts Lemma 6.3.2.

26. Page 80, line +1.

The vertices x and y are not defined. I guess, $y \in \mathrm{Fix}(f)$ and $x \in \mathrm{Fix}(g)$.

27. Page 84, lines +3 – +5.

It seems that H must be replaced by any torsion subgroup of H here.

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