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Report on the Doctoral Thesis "Canonical quotients in model theory" by Adrián Portillo Fernández

Portillo's thesis is an important and strong contribution to the rapidly emerging area on the intersection of topological dynamics and model theory that studies actions of definable and automorphism groups of first-order structures on certain spaces of types (i.e. ultrafilters on the Boolean algebra of definable subsets), with the focus on the existence and properties of various canonical quotients and Shelah's classification theory, as well as connections to special classes of dynamical systems.

The interest in such quotients stems from several closely related lines of research. On the one hand, it is motivated by the influential conjecture of Pillay on groups in *o*-minimal structures, eventually proved by Hrushovski, Peterzil and Pillay, and its analogs and generalizations in other model theoretically tame settings. The class of NIP theories (i.e., theories with No Independence Property) was introduced by Shelah in his work on the classification program. A theory T is *NIP* if for every partitioned formula $\phi(x, y)$, there is no model of T in which we can find tuples $(a_i : i \in \omega)$ and $(b_s : s \subseteq \omega)$ such that $\phi(a_i, b_s)$ holds if and only if $i \in s$. Typical examples of NIP theories are stable and *o*-minimal theories, and in recent years NIP theories have become a very active domain of study in their own right. A fundamental theorem of Shelah demonstrates that every saturated NIP group G admits the smallest type-definable subgroup of bounded index, i.e. of index smaller than the saturation, denoted G^{00} (which can be thought of as an abstract analogue of the subgroup of "infinitesimals"). The quotient G/G^{00} is naturally equipped with the structure of a compact topological group, with respect to the logic topology. In the *o*-minimal case, it is in fact a compact Lie group. Pillay's conjecture predicted that if G is a definably compact group in a saturated *o*-minimal expansion of a field, then the dimension of G/G^{00} as a Lie group equals the dimension of G as a definable set in an *o*-minimal structure — a surprising connection between the pure lattice of definable subsets of groups in *o*-minimal theories and real Lie groups, which created an active subarea. On the other hand, many methods and objects from topological dynamics we introduced into the picture, starting with the work of Newelski. In particular, deep connections between NIP groups and *tame* dynamical systems as studied by Glasner, Megrelishvili and others have emerged, and play an important role in Portillo's work (as well as the folklore connection between stable groups and WAP dynamical systems). In certain settings, the aforementioned canonical model theoretic quotients correspond precisely to the classical topological invariants of the associated dynamical systems, and understanding the extent of this interplay is currently an active research direction. Finally, let us mention another source of interest in such quotients: the study of hyperimaginaries is central in the amalgamation problems in simple theories, and whether they can be eliminated remains a central open problem.

Portillo's thesis makes important and innovative contributions to each of these directions, in particular answering some open questions by the experts in the area, and develops a coherent and promising program of systematically studying tame quotients (providing a very natural analogy to the study of WAP and tame factors in the decompositions of dynamical systems), raising many exciting directions for future research. Below I discuss and evaluate the content of the thesis in greater detail.

Chapter 1 is an introduction summarizing the main results of the thesis.

Chapter 2 contains a presentation of the necessary preliminaries concerning the relevant results from pure model theory, continuous model theory and hyperdefinable sets, structural Ramsey theory and topological dynamics. Some of the results collected here and presented in a concise and efficient manner, are spread out through a number of sources in the literature and are often hard to pin down for an exact reference, hence in my opinion this chapter could serve as a quick and useful reference guide for a person entering the area. In particular, this includes a presentation of the folklore connection between hyperimaginaries and continuous logic imaginaries, as well as the correspondence between the model theoretic properties of stability and NIP, and the notions of WAP and tame systems in topological dynamics. Additionally, the appendix contains a proof that the properties of stability and NIP for hyperdefinable sets are preserved under (possibly infinite) Cartesian products.

Chapter 3 studies maximal stable quotients of type-definable groups in NIP theories, and is published (jointly with his advisor) as "On Stable Quotients", Notre Dame Journal of Formal Logic 63.3 (2022). The maximal bounded quotients of NIP groups discussed above can be thought of as "trivial" from the Shelah classification point of view, as compactness of a hyperimaginary sort (or a continuous structure) should be viewed as corresponding to finiteness in classical logic. It then becomes a very natural (and surprisingly little considered up to this point) question to understand if "finiteness" can be relaxed to stability or NIP, and if maximal quotients with the corresponding property still exist. Earlier Pillay and Haskell, generalizing Shelah's theorem, established existence of maximal stable quotients of groups type-definable in NIP theories, i.e. existence of G^{st}, the smallest type-definable subgroup with G/G^{st} stable. Here Portillo solves two problems that they left open. The first result shows that if G is a type-definable group in a distal theory (i.e. a "purely unstable" NIP theory), then $G^{st} = G^{00}$. This is approached by systematically developing stability and distality of hyperdefinable sets using continuous logic, and proving that distality is preserved under passing from a theory T to the hyperimaginary expansion T^{heq} (generalizing some results of Simon). The second main result of the chapter provides an example of a group G definable in a non-distal, NIP theory so that $G = G^{00}$ but G^{st} is not an intersection of definable groups (in a saturated extension of $(\mathbb{R}, +, [0, 1])$). This chapter also includes some curious observations concerning the existence of such groups of finite exponent.

Chapter 4 studies the existence of finest relatively type-definable equivalence relations on invariant types with stable quotients (which can be viewed as the analog of the previous question for actions of the automorphism groups), and is based on the paper by Krzysztof Krupiński and Adrián Portillo. "Maximal stable quotients of invariant types in NIP theories", The Journal of Symbolic Logic (2023). The main result establishes that for an NIP theory T , a sufficiently saturated model $\mathbb{C}$ of T and an invariant global type p, there exists a finest relatively type-definable over a small set of parameters from $\mathbb{C}$ equivalence relation E^{st} on the set of realizations of p which has stable quotient. The proof adapts (in a highly non-trivial manner) the ideas of the proof of the existence of G^{st} (which in turn adapted Shelah's proof of the existence of G^{00}), but working with relatively type-definable subsets of the group of automorphisms of the monster model from earlier work of Hrushovski, Krupiński and Pillay. Interestingly, this proof uses very little about stability, and could probably be adapted to some other tameness conditions within NIP, e.g. to higher order analogs of stability such as NFOP_2, etc. The chapter also discusses the subtlety of the transfer of the existence of finest relatively type-definable equivalence relations with stable quotients between models, and provides an interesting and detailed computation of E^{st} in two concrete examples (expansions of local orders).

Chapter 5 is dedicated to a higher arity generalization of NIP, the n-dependence property, in the context of hyperimaginary sets, and is based on the solo preprint by Portillo, "N-dependent continuous theories and hyperdefinable sets", arXiv:2405.19830. It considers the question of the existence of generalized indiscernibles (indexed not by a linear order, but by some more complicated structures) in continuous logic, showing that a first-order structure has the continuous modelling property if and only if its age has the embedding Ramsey property. Generalized indiscernibles are then used to characterize n-dependence for continuous theories and first-order hyperdefinable sets in terms of the collapse of indiscernible sequences. This provides a basis for possible future investigation of the question of relative absoluteness of G^{st}, by analogy with the relative absoluteness of G^{00}, in n-dependent theories (which in the 2-dependent case says that $G^{00}_{MA} = G^{00}_M \cap G^{00}_A \cap G^{00}_{M_0 A}$ for some M_0 submodel of M of absolutely bounded size, where M is sufficiently saturated with respect to A).

Chapter 6, to appear as a joint preprint of Portillo with Krupiński, investigates maximal WAP and tame quotients of the space of types $S_X(\mathbb{C})$ under the action of the automorphism group, for a sufficiently saturated model $\mathbb{C}$ and a type-definable set X. Namely, there exists F_{WAP} a finest closed invariant equivalence relation on $S_X(\mathbb{C})$ such that the flow $(Aut(\mathbb{C}), S_X(\mathbb{C})/F)$ is WAP. The main result of the section demonstrates that the ideal Ellis group of this flow (as a topological group with τ-topology) does not depend on the monster model $\mathbb{C}$, hence truly is an invariant of the theory T. An analogous result is obtained for the finest tame equivalence relation, as well as for the stable and NIP counterparts (which contain strictly less information in general). All of these are deduced from a more general theorem on absoluteness of the Ellis group for quotients by compatible equivalence relations living on different models, which is likely to apply to other "tame" quotients as well. These results are proved using methods from topological dynamics, and the chapter also contains a nice concise exposition (following Simon and Krupiński) connecting the absoluteness (i.e. independence of the monster model $\mathbb{C}$ in which it is computed) of the Ellis group of the action of the automorphism group on the space of global types from earlier work of Krupiński, Newelski and Simon, and Hrushovski's work on definability patterns.

Let me summarize. Portillo's thesis contains novel and important results. It develops a rather convincing and fundamental general theory of tame quotients, relying on the systematic use of continuous logic and tools from topological dynamics. On the one hand, the methods developed vastly generalize some of the existing results and techniques, and answer several questions left open by the leading researchers in this competitive area. On the other hand, objects and methods developed in the thesis form a beginning of an exciting research program, and undoubtedly will lead to a lot of future work and continue to grow in impact. Two chapters are already published as two papers in leading journals in mathematical logic, and I do not doubt that the remaining two chapters will also be published in high quality logic or even general mathematical journals. The presentation is well organized, (mostly) clear and engaging, there are a few linguistic and minor mathematical inaccuracies, as well as some details missing, that I will communicate to Portillo directly, but none of them affect the results.

Based on all of this, I very strongly recommend acceptance of Portillo's thesis and I suggest that it should be considered for the nomination for a doctoral dissertations award.

Sincerely,

Artem Chernikov
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and
Professor of Mathematics, University of California Los Angeles (on leave)