University of Wrocław
Faculty of Mathematics and Computer Science
Mathematical Institute

Uniwersytet Wrocławski
Wydział Matematyki i Informatyki
Instytut Matematyczny

Daria Perkowska

* operation

Operacja *

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dr hab. Piotr Borodulin-Nadzieja

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dr. hab. Piotra Borodulina-Nadziei

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Contents

0 Introduction

My thesis is devoted to the $\sigma - ideals$ of strongly measure sets, strongly meager sets and star operation. The study of those notions has quite a long tradition involving many mathematicians connected to Wrocław (e.g. Mycielski, Pawlikowski, Sabok, Seredyński and Solecki). We present several basic and well known results concerning those notions. Also, we show certain new result which is a rather direct generalization of a theorem proved in a recent paper of Horbaczewska and Lindner [11]. It partially answers a question asked by Seredyński in [4]

In Section 1 we introduce the basic notions used in the thesis, in particular we define cardinal coefficients: additivity, covering and cofinality of an ideal. Section 2 is devoted to Martin's Axiom, the well-known axiom which implies that the coefficients mentioned above are of cardinality $\mathfrak{c}$ (this assumption will be needed in the last section). In Section 3 we define the families of strong measure zero and strongly meager sets, we briefly overview its history and we prove some classical results:

- the family of strong measure zero sets is a σ-ideal (3.1.2),
- all countable sets are strong measure zero and strongly meager (3.1.5, 3.2.3),
- there is a null set which is not strong measure zero and a meager set which is not strongly meager (3.1.4, 3.2.2),
- assuming Continuum Hypothesis, we prove that there is an uncountable strong measure zero set (Luzin set) and we mention the example of an un-

countable strongly meager set (Sierpiński set).

In Section 4 we prove some basic facts on the star operation. Finally we prove that $\text{cov}(\mathcal{M}) = \mathfrak{c}$ (and so, Martin's Axiom) implies $\mathcal{M} = \mathcal{M}^{**}$ and that $\text{cov}(\mathcal{N}) = \mathfrak{c}$ (and so, Martin's Axiom) implies that $\mathcal{N} = \mathcal{N}^{**}$. The proof is based on the proof of Horbaczewska and Lindner from [11] when they prove it under Continuum Hypothesis.

We finish the thesis with a natural open problem. We were not able to solve it.

1 Preliminaries

Definition 1.1. We say that a subset $B \subseteq X$ of a topological space $X$is **nowhere dense** if its closure has empty interior.

Equivalently B is **nowhere dense** in X if for each open set $U \subseteq X$, the set $B \cap U$ is not dense in U.

Definition 1.2. A subset $A \subseteq X$ of a topological space X is called **meager** if it is a countable union of nowhere dense subsets of X.

Definition 1.3. We say that a subset $\mathcal{I} \subseteq P(X)$ is a **ideal** if the following properties are satisfied:

- $\emptyset \in \mathcal{I}$,
- When $A \in \mathcal{I}$ and $B \subseteq X$, then $B \subseteq A \implies B \in \mathcal{I}$,
- If $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$.

Definition 1.4. We say that a subset $\mathcal{I} \subseteq P(X)$ is a **σ-ideal** if it is an ideal and moreover

- If $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{I}$ then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{I}$.

Now we can introduce some σ-ideals of real line.

- σ-ideal of meager sets of real line we will denote by $\mathcal{M}$.
- σ-ideal of Lebesgue measure zero sets of real line we will denote by $\mathcal{N}$.
- σ-ideal of countable sets of real line we will denote by $Count$.

Definition 1.5. We define **additivity** of the ideal $\mathcal{I} \subseteq X$ as the smallest number of sets from $\mathcal{I}$ whose union is not in $\mathcal{I}$ anymore. Formally,

$$\text{add}(\mathcal{I}) = \min\{|A| : A \subseteq \mathcal{I} \wedge \bigcup A \notin \mathcal{I}\}.$$

Definition 1.6. We define **covering** of the ideal $\mathcal{I} \subseteq X$ as the smallest number of sets from $\mathcal{I}$ whose union is all of X. Formally,

$$\operatorname{cov}(\mathcal{I}) = \min\{|A| : A \subseteq \mathcal{I} \wedge \bigcup A = X\}.$$

Definition 1.7. We define the **cofinality** of the ideal $\mathcal{I}$ as the least cardinality of such a subfamily of $\mathcal{I}$ that every element of $\mathcal{I}$ can be bounded from above by an element of that subfamily. Formally,

$$\operatorname{cof}(\mathcal{I}) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{I} \wedge (\forall A \in \mathcal{I})(\exists B \in \mathcal{B})(A \subseteq B)\}.$$

It is obvious that $\operatorname{add}(\mathcal{M}) \leqslant \mathfrak{c}$, $\operatorname{add}(\mathcal{N}) \leqslant \mathfrak{c}$, $\operatorname{cov}(\mathcal{M}) \leqslant \mathfrak{c}$ and $\operatorname{cov}(\mathcal{N}) \leqslant \mathfrak{c}$.

Definition 1.8. We say that a subset of a topological space is an G_δ **set** if it is a countable intersection of open sets.

Definition 1.9. We say that a subset of a topological space is an F_σ **set** if it is a countable union of closed sets.

Proposition 1.10. $\operatorname{cof}(\mathcal{M}) \leqslant \mathfrak{c}$

Proof. Since we know that every meager set we can cover with F_σ set which is meager we get $\operatorname{cof}(\mathcal{M}) \leqslant \mathfrak{c}$.

□

Proposition 1.11. $\operatorname{cof}(\mathcal{N}) \leqslant \mathfrak{c}$

Proof. Since we know that every measure zero set we can cover with G_δ set which has measure zero we get $\operatorname{cof}(\mathcal{N}) \leqslant \mathfrak{c}$.

□

2 Martin's Axiom

In this section we will introduce Martin's Axiom, the standard axiom which makes the coefficients defined above big.

We will start with the Rasiowa-Sikorski Lemma. Later we can generalise it to Martin's Axiom. We have to start with some definitions.

Definition 2.1. We say that a subset D in a poset $\mathcal{P}$ is **dense** if

$$(\forall x \in \mathcal{P})(\exists q \in D)(q \leqslant x).$$

Definition 2.2. W say that $\mathcal{F} \subseteq \mathcal{P}$ is a **filter**, if the following conditions hold:

- $\mathcal{F} \neq \emptyset$,
- $(p \in \mathcal{F} \wedge p' \geqslant p) \implies p' \in \mathcal{F}$,
- $(p, q \in \mathcal{F}) \implies (\exists r \leqslant p, q)\ r \in \mathcal{F}$.

Definition 2.3. We say that a filter $\mathcal{G}$ is **$\mathcal{D}$-generic** if for a dense $\mathcal{D} \subseteq \mathcal{P}$ it holds that $\mathcal{G} \cap \mathcal{D} \neq \emptyset$.

Lemma 2.4 (Rasiowa-Sikorski)**.** *Let $\mathcal{P}$ be a partial order and $\mathcal{D}$ a countable family of dense subsets of $\mathcal{P}$. Then there exists a $\mathcal{D}$-generic filter in $\mathcal{P}$.*

Proof. Enumerate $\mathcal{D} = \{D_0, D_1, ...\}$. We take any $d_0 \in D_0$. The, using the fact that D_1 is dense, we take $d_1 \in D_1$ such that $d_1 \leqslant d_0$. We continue in the same way receiving a decreasing sequence of elements, each of them is from the next set from family. From this sequence we obtain a filter $\mathcal{G} = \{p : \exists n, p \geqslant d_n\}$.

□

Now we will need another definitions to upgrade the Rasiowa-Sikorski Lemma.

Definition 2.5. We say that elements $p, q \in A$ are **incompatible** if there don't exists $r \in A$ such that $r \leqslant p$ and $r \leqslant q$.

Definition 2.6. A set $A \subseteq \mathcal{P}$ is an **antichain** in $\mathcal{P}$ if every two different elements $p, q \in A$ are incompatible.

Definition 2.7. We say that a partial order $\mathcal{P}$ satisfies **ccc** if all the antichains are countable.

Now we can define Martin's Axiom.

Definition 2.8. Let κ be a cardinal number. By $MA(\kappa)$ we denote the following axiom: Let $\mathcal{P}$ be a partial order that satisfies the *ccc* and let $\mathcal{D}$ be a family of dense subsets of $\mathcal{P}$ with $|\mathcal{D}| \leqslant \kappa$. Then there exists a $\mathcal{D} - generic$ on $\mathcal{P}$.

Definition 2.9. Martin's Axiom says that $MA(\kappa)$ holds for every $\kappa < 2^{\aleph_0}$.

Note that $MA(\aleph_0)$ is equivalent to Rasiowa-Sikorski Lemma.

Theorem 2.10. *Continuum Hypothesis implies Martin's Axiom.*

Proof. Since the Continuum Hypothesis holds we know that $2^{\aleph_0} = \aleph_1$. So the only $MA(\kappa)$ required for MA to be true is $MA(\aleph_0)$. It's obvious from previous remark.

□

Theorem 2.11. *Assume Martin's Axiom.* $\text{add}(\mathcal{N}) = \mathfrak{c}$

Proof. The proof follows the proof in [2].

Let $\{E_i : i \in I\}$ be the $\leqslant \kappa$ sets of measure 0, and let $E = \bigcup E_i$. Let $\varepsilon > 0$. We must find an open set including E which has measure $< \varepsilon$.

Let P be the collection of all open sets of measure $< \varepsilon$, and $p \leqslant q$ mean $p \subseteq q$. We have to show that the poset P satisfies *ccc*. Let Q be pairwise incompatible subset of P. Let $Q_n = \{p \in Q : \mu(p) \leqslant (1 - 2^{-n}) \cdot \varepsilon\}$. Now we show that Q_n is countable for each n which is enough to have *ccc*.

For each $p \in Q_n$, choose $\bar{p} \subseteq p$, so that $\bar{p}$ is a finite union of open intervals with rational endpoint and $\mu(p - \bar{p}) < 2^{-n} \cdot \varepsilon$. Since there are only countable many such finite unions, we have only to show that if p and q are distinct members of Q_n, then $\bar{p} \neq \bar{q}$.

Suppose $\bar{p} = \bar{q}$. Then $p \cup q \subseteq (p - \bar{p}) \cup q$, so $\mu(p \cup q) < 2^{-n} \cdot \varepsilon + (1 - 2^{-n}) \cdot \varepsilon = \varepsilon$. Then we have that p and q are compatible, contradicting $p, q \in Q$.

For $i \in I$, let $D_I = \{p \in P : E_i \subseteq p\}$. Using $\mu(E_i) = 0$, we see that D_i is dense in P. From Martin's Axiom we know that there is $D - \textit{generic}$ which meets every D_i.

Let G be the union of the members of Q. Then G is open. From $D_i \cap Q \neq \emptyset$ we find that $E_i \subseteq G$, so $E \subseteq G$.

Assume that $\mu(G) > \varepsilon$. G is a countable union of sets in Q. It follows that there is finite union G_1 of sets in Q such that $\varepsilon \leqslant \mu(G_1)$. Since Q is D-generic, some member of Q includes G_1 and hence has measure $\varepsilon \leqslant$. This is a contradiction, because elements of Q has measure $< \varepsilon$.

□

3 Strong measure zero sets and strongly meager sets

In this section we will define the notion of strongly measure zero sets (coined by Borel in 1919) and a much younger notion of strongly meager sets.

3.1 Strongly measure zero sets

Definition 3.1.1. A set $A \subseteq \mathbb{R}$ has **strong measure zero** when for every sequence (ε_n) of positive reals there exists a sequence (I_n) of intervals, such that $|I_n| \leqslant \varepsilon_n$ and A is contained in the union of the I_n.

- σ-ideal of strong measure zero sets of real line we will denote as $\mathcal{SMZ}$.

Proposition 3.1.2. *$\mathcal{SMZ}$ is $\sigma-$ideal of real line.*

Proof. It's clear that $\emptyset \in \mathcal{SMZ}$ and that a subset of strong measure zero set has strong measure zero. So we have to show that if $\{X_n\}_{n\in\mathbb{N}} \subseteq \mathcal{SMZ}$ then $\bigcup_{n\in\mathbb{N}} X_n \in \mathcal{SMZ}$.

Let $\{X_n : n \in \omega\}$ be a family of strong measure zero sets. We will show that $\bigcup_{n\in\omega} X_n$ is a strong measure zero set. Given $\langle\varepsilon_n : n \in \omega$ let $\{A_n : n \in \omega\}$ be any family of pairwise disjoint, infinite subsets of ω. For each $m \in \omega$ there exists $\{x_n^m : n \in \omega\}$ such that $X_m \subseteq \bigcup_{n\in A_n}(x_n^m - \varepsilon_n, x_n^m + \varepsilon_n)$. Now we have

$$X \subseteq \bigcup_{m\in\omega} \bigcup_{n\in A_n} (x_n^m - \varepsilon_n, x_n^m + \varepsilon_n)$$

□

First, we will prove an obvious fact that $\mathcal{SMZ}$ is contained in $\mathcal{N}$.

Proposition 3.1.3. $\mathcal{SMZ} \subseteq \mathcal{N}$

Proof. Take $\varepsilon > 0$ and $\varepsilon_n = \frac{\varepsilon}{2^{n+1}}$ So there exists a collection of intervals I_n such that $A \subseteq \bigcup I_n$, so $\lambda(A) \leqslant \sum \frac{\varepsilon}{2^{n+1}} = \varepsilon$. □

The ideal of strongly measure zero sets is however much smaller than $\mathcal{N}$.

Proposition 3.1.4. *There exists a measure zero set which does not have strong measure zero, so $\mathcal{N} \neq \mathcal{SMZ}$*

Proof. Cantor set is an example of a set of measure zero that don't have strong measure zero.

Suppose that $I_1.I_2, ...$ are open intervals in $\mathbb{R}$ such that I_n has length 3^{-n}

Note that I_1 must be disjoint from $[0, \frac{1}{3}]$ or $[\frac{2}{3}, 1]$. Label the interval disjoint from I_1 by A_1

Now I_2 must be disjoint from one of two closed intervals obtained from deleting the open middle third of A_1, Label that one A_2.

We continue like this. For each n the interval I_{n+1} from one of the closed intervals obtained from deleting the middle third of A_n, and we label one of this A_{n+1}

We have a decreasing sequence $A_1 \supseteq A_2 \supseteq ...$ of nonempty compact subsets of $\mathbb{R}$, and so their intersection must be nonempty. Any point of this intersection must belong to Cantor set and it's not in $\bigcup_{i=1} I_n$ so the sequence of open intervals we began with can't cover the Cantor set. $\square$

The ideal $\mathcal{SMZ}$ contains all the countable subsets of the real line.

Proposition 3.1.5. *Count* $\subseteq \mathcal{SMZ}$

Proof. We know that every countable set is countable union of singletons.
Every singleton we can cover with an interval $|I| \leqslant \varepsilon$ for all $\varepsilon > 0$. So we can cover a countable set with a sequence of intervals (I_n), such that $|I_n| \leqslant \varepsilon_n$ for all $\varepsilon_n > 0$. $\square$

It is not clear if it contains anything more than countable sets. The Borel Conjecture says that

$$\mathcal{SMZ} = Count.$$

It was known that consistently one can create uncountable sets which are strongly measure zero. We will present now the simplest example of this sort.

Definition 3.1.6. A subset L of $\mathbb{R}$ is called a **Luzin set** if L is uncountable, but for every nowhere dense subset K of $\mathbb{R}$ the intersection $K \cap L$ is countable.

Proposition 3.1.7. *Every Luzin set has strong measure zero.*

Proof. Proof follows as in [7]

Let X be a Luzin set. Take a sequence of positive reals $\langle \varepsilon_n : n \in \omega \rangle$, and let $\langle q_n : n \in \omega$ be an enumeration of all rationals.

Since $\bigcup_{n\in\omega}(q_n - \varepsilon_{2n}, q_n - \varepsilon_{2n})$ is a comeager set, it follows that

$$X \backslash \bigcup (q_n - \varepsilon_{2n}, q_n - \varepsilon_{2n})$$

is a countable set, name it $\{x_n : n \in \omega\}$. Then

$$X \subseteq \bigcup (q_n - \varepsilon_{2n}, q_n - \varepsilon_{2n}) \cup \bigcup (x_n - \varepsilon_{2n+1}, x_n - \varepsilon_{2n+1})$$

□

Theorem 3.1.8. *Continuum Hypothesis implies existence of Luzin set.*

Proof. Proof follows as in [6].

We can define Luzin set as an uncountable set of reals that has countable intersection with every closed nowhere dense set of reals. We know that there are exactly $2^{\aleph_0}$ nowhere dense closed subsets of $\mathbb{R}$. Assume CH holds, and let $\langle K_\xi : \xi < \omega_1 \rangle$ be an enumeration of all closed nowhere dense subset of $\mathbb{R}$.

Construct $L = \{x_\eta : \eta < \omega_1\}$ in such way that $x_\eta \in \mathbb{R} \backslash (\bigcup_{\xi<\eta} K_\xi \cup \{x_\xi : \xi < \eta\})$ for all $\eta < \omega_1$. This is possible since the real line is not a union of countably many nowhere dense subsets. Clearly, L is uncountable. If K is a nowhere dense subset of $\mathbb{R}$, then $cl(K) = K_\xi$ for some $\xi < \omega_1$, and $L \cap K \subseteq \{x_\eta : \eta \leqslant \xi\}$. It follows that L is Luzin set.

□

Proposition 3.1.9. *Continuum Hypothesis implies existence of uncountable strong measure zero set and so it implies the failure of Borel Conjecture.*

Proof. We know that Continuum Hypothesis implies existence of Luzin set and we know that every Luzin set has strong measure zero. So we have uncountable strong measure zero set. □

However, for many years it was open if Borel Conjecture is consistent with the usual axioms of set theory. It was solved by Laver in [3]

One of the most astonishing theorems in the theory of strongly measure sets is the following, due to Galvin, Mycielski and Solovay (see [1]).

Theorem 3.1.10. *A set $X \subseteq \mathbb{R}$ is strongly measure zero if and only if for every meager set H it holds that $X + H \neq \mathbb{R}$.*

This theorem motivates the notion of strongly meager sets which we examine in the next section.

3.2 Strongly meager sets.

Definition 3.2.1. A set $X \subseteq \mathbb{R}$ is **strongly meager** if for every measure zero set H it holds that $X + H \neq \mathbb{R}$.

We will denote the family of strongly meager sets of real line by $\mathcal{SM}$.

An example of a set that it is not strongly meager is $[0, 1]$.

Proposition 3.2.2. $\mathcal{M} \nsubseteq \mathcal{SM}$

Proof. An example of a set that has measure zero but is not in $\mathcal{SM}$ is the Cantor set, but copied in every interval $[n, n + 1]$ for any $n \in \mathbb{Z}$.

To show the above, it is enough to show that $C+C = [0, 2]$, where C is the Cantor set.

We know that the Cantor set is the set of all numbers between 0 and 1 that can be written in base 3 using only the digits 0 and 2. So we have to show that we can get any number between 0 and 2 as a sum of two numbers that written in base 3 have only the digits 0 and 2.

Let c be any number from $[0, 2]$. We want to have $c = a+b$, where $a \in C$ and $b \in C$. By a_i we denote the digit on the i-th place in the ternary (base 3) expansion on the number a. Same for b and c. We can get our a and b according to the algorithm below.

1. If the only digits occurring in c_i are 0 or 2, then let $a = 0$ and $b = c$.
2. When we have first 1 in our cycle in place number i as a_i we take 2 and as b_i we take 2. If $c_{i+1} = 1$ we take $a_{i+1} = b_{i+1} = 0$ and we go to step number one and start our cycle again. If $c_{i+1} = 0$ we take $a_{i+1} = 0$ and $b_{i+1} = 2$, if $c_{i+2} = 2$ we take $a_{i+2} = 2$ and $b_{i+2} = 2$. We do this steps as long as $c_{i+n} \neq 1$. When $c_{i+n} = 1$ we take $a_{i+n} = 0$ and $b_{i+n} = 0$ and go to step 1.

We see that with this algorithm we can get every number from $[0, 2]$. □

Notice that the above proof would work also to show that there is a null set which is not strongly measure zero (if we use Galvin-Mycielski-Solovay theorem).

Proposition 3.2.3. $Count \subseteq \mathcal{SM}$

Proof. We have to show that $C + H \neq \mathbb{R}$ for all $H \in \mathcal{N}$ and $C \in Count$. We know that $\mathcal{N}$ is translation invariant so if we take any $r \in \mathbb{R}$ we have

$$(\forall H \in \mathcal{N})(H + r \neq \mathbb{R})$$

Since $\mathcal{N}$ is $\sigma - ideal$ we can "move" our set by countable many singletons.
So we have $Count \subseteq \mathcal{SM}$.

□

Definition 3.2.4. A subset S of $\mathbb{R}$ is called a **Sierpiński set** if S is uncountable, but for every $N \subseteq \mathbb{R}$ of Lebesgue measure zero the intersection $N \cap S$ is countable.

In [5] Pawlikowski showed that every Sierpiński set is strongly meager, answering a famous open problem of Galvin.

Theorem 3.2.5. *Continuum Hypothesis implies existence of Sierpiński set.*

Proof. Take a collection of $2^{\aleph_0}$ measure zero sets of $\mathbb{R}$ such that every measure zero set is contained in one of them. Assume CH holds and let $\langle N_\xi : \xi < \omega_1 \rangle$ be an enumeration of our collection.

Construct $S = \{x_\eta : \eta < \omega_1\}$ such that $x_\eta \in \mathbb{R} \setminus (\bigcup_{\xi<\eta} N_\xi \cup \{x_\xi : \xi < \eta\})$ for all $\eta < \omega_1$. This is possible since the real line is not a union of countably many measure zero sets. Clearly, S is uncountable and $S \cap N$ is countable. It follows that S is Sierpiński set. □

The dual Borel Conjecture says that

$$\mathcal{SM} = Count.$$

Proposition 3.2.6. *Continuum Hypothesis implies existence of uncountable strongly meager set and consequently it implies the failure of dual Borel conjecture.*

Proof. We know that Continuum Hypothesis implies existence of Sierpiński set and we know that every Sierpiński set is strongly meager. So we have uncountable strongly meager set.

□

For many years the problem if Borel Conjecture and Dual Borel Conjecture holds simultaneously was open. It was solved by Goldstern, Kellner, Shelah and Wohofsky in [10].

4 * operation

The Galvin-Mycielski-Solovay theorem and the notion of strongly meager sets motivate the following abstract notion.

Let $(X, +)$ be an abelian group. For $A, B \subseteq X$ we write

$$A + B = \{a + b : a \in A, b \in B\}.$$

Definition 4.1. For a family $\mathcal{F} \subseteq \mathcal{P}(X)$ let:

$$\mathcal{F}^* = \{A \subseteq X : \forall_{F \in \mathcal{F}} A + F \neq X\}.$$

We write $\mathcal{F}^{**} = (\mathcal{F}^*)^*$ and $\mathcal{F}^{*(n+1)} = (\mathcal{F}^{*(n)})^*$

In this sense the Galvin-Mycielski-Solovay says that $\mathcal{SMZ} = \mathcal{M}^*$ and the sets of strongly meager sets can be defined as $\mathcal{SM} = \mathcal{N}^*$.

Now, before we reach the main results of the thesis, we prove several simple general facts about the star operation.

4.1 General remarks on *.

Proposition 4.1.1. $\mathcal{G} \subseteq \mathcal{F}^* \iff \mathcal{F} \subseteq \mathcal{G}^*$

Proof. ($\Longrightarrow$) Assume we have $\mathcal{G} \subseteq \mathcal{F}^*$ From definition $\mathcal{F}^* = \{A \subseteq X : \forall_{F \in \mathcal{F}} A + F \neq X\}$. Since $\mathcal{G}$ is subset of $\mathcal{F}^*$ we have that $\mathcal{G} \subseteq \bigcup A$. We know that $\forall_{A \in \mathcal{F}^*} \forall_{F \in \mathcal{F}} A + F \neq X$, so since $\mathcal{G} \subseteq \bigcup A$ we have $\mathcal{F} \subseteq \mathcal{G}^*$

($\Longleftarrow$) The converse follows analogically. $\square$

Proposition 4.1.2. $\mathcal{F} \subseteq \mathcal{F}^{**}$

Proof. By the definition $\mathcal{F}^{**} = \{A \subseteq X : \forall_{F \in \mathcal{F}^*} A + F \neq X\}$. So we have to show that $\forall_{F \in \mathcal{F}} \forall_{A \in \mathcal{F}^*} A + F \neq X$. From definition $\mathcal{F}^*$ we know that in $\mathcal{F}^*$ are only sets A such that $A + F \neq X$ what we had to show. $\square$

Proposition 4.1.3. $\mathcal{G} \subseteq \mathcal{F} \Longrightarrow \mathcal{F}^* \subseteq \mathcal{G}^*$

Proof. Assume $\mathcal{G} \subseteq \mathcal{F}$. From definition $\mathcal{F}^* = \{A \subseteq X : \forall_{F \in \mathcal{F}} A + F \neq X\}$. Since $\mathcal{G} \subseteq \mathcal{F}$ if $A \in \mathcal{F}^*$ and $A + F \neq X$ for $F \in \mathcal{F}$ then we know that $A \in \mathcal{F}^*$, $A + G \neq X$ for $G \in \mathcal{G}$. $\square$

Proposition 4.1.4. $\mathcal{F}^*$ *is closed under taking subsets and translation invariant*

Proof. Take $A \in \mathcal{F}^*$. We know that $\forall_{F \in \mathcal{F}} A + F \neq X$. But taking any subset $B \subseteq A$ we have $\forall_{F \in \mathcal{F}} B + F \neq X$, so $\mathcal{F}^*$ is closed under taking subsets.

Take $A \in \mathcal{F}^*$. We know that $\forall_{F \in \mathcal{F}} A + F \neq X$. So there exists $a \in X$ such that $a \notin A + F$ for all $F \in \mathcal{F}$. Taking a translation $A + x$ we get an element $a + x$ such that $a + x \notin (A + x) + F$ for all $F \in \mathcal{F}$. So $\mathcal{F}^*$ is translation invariant. $\square$

Proposition 4.1.5. $\mathcal{F}^{*(n+2)} = \mathcal{F}^{*(n)}$ *for* $1 \leqslant n$

Proof. From 4.1.2 and 4.1.3 we have that $\mathcal{F}^{***} \subseteq \mathcal{F}^{*}$ and $\mathcal{F}^{**} \subseteq \mathcal{F}^{****}$. So we know that $\mathcal{F}^{*(2n+1)} \subseteq \mathcal{F}^{*(2n-1)}$ and $\mathcal{F}^{*(2n)} \subseteq \mathcal{F}^{*(2n+2)}$. Now he have the reverse inclusion. First we have to proof that $\mathcal{F}^{*} \subseteq \mathcal{F}^{***}$

From definition $\mathcal{F}^{***} = \{A \subseteq X : \forall_{F \in \mathcal{F}^{**}} A + F \neq X\}$. So we have to show that $\forall_{F \in \mathcal{F}^{*}} \forall_{A \in \mathcal{F}^{**}} A + F \neq X$. From definition $\mathcal{F}^{**}$ we know that in $\mathcal{F}^{**}$ are only sets A such that $A + F \neq X$ what we had to show.

From 4.1.3 we have $\mathcal{F}^{****} \subseteq \mathcal{F}^{**}$ and $\mathcal{F}^{***} \subseteq \mathcal{F}^{*****}$. So $\mathcal{F}^{*(2n-1)} \subseteq \mathcal{F}^{*(2n+1)}$ and $\mathcal{F}^{*(2n+2)} \subseteq \mathcal{F}^{*(2n)}$. □

Proposition 4.1.6. *If* $\mathcal{I} \subseteq \mathcal{P}(\mathbb{R})$ *is translation invariant* $\sigma - ideal$*, then* $Count \subseteq \mathcal{I}^{*}$

Proof. Since $\mathcal{I} \subseteq \mathbb{R}$ is translation invariant we know that $(\forall I \in \mathcal{I})(I + r \in \mathcal{I})$.
In $\mathcal{I}^{*}$ are all singletons, because $(\forall I \in \mathcal{I})(I + r \neq \mathbb{R})$.

But since $\mathcal{I}$ is a σ-ideal we know that we can "move" our sets by countable many singletons. So in $\mathcal{I}^{*}$ are all countable sets.

□

Theorem 4.1.7. $Count^{*}$ *is the union of all proper, translation invariant* σ*-ideals of subsets of* X

Proof. Let $\mathcal{J}$ be a proper, translation invariant $\sigma - ideal$ of subsets of X and let $I \in \mathcal{J}$. The existence of a countable set $C = \{c_1, c_2, ...\}$ such that $I + C = X$ is impossible since $I + C = \bigcup_{k \in \mathbb{N}} (I + c_k) \in \mathcal{J}$, so $I \in Count^{*}$.

For any $A \in Count^{*}$, the family of sets $\mathcal{J}_A = \{P \subseteq X : \exists (C \in Count)(P \subseteq A+C)\}$ is a translation invariant proper σ-ideal containing set A

□

Proposition 4.1.8. *As a consequence of previous theorem we have*

$$\mathcal{M} \neq Count^{*},$$
$$\mathcal{N} \neq Count^{*}.$$

Proposition 4.1.9. $\mathcal{N} \subseteq Count^{*}$

Proof. We know from 4.1.6 that $Count \subseteq \mathcal{N}^*$. Using Theorem 4.1.1 we have $\mathcal{N} \subseteq Count^*$

□

Proposition 4.1.10. $\mathcal{M} \subseteq Count^*$

Proof. We know from 4.1.6 that $Count \subseteq \mathcal{M}^*$. Using Theorem 4.1.1 we have $\mathcal{M} \subseteq Count^*$

□

Proposition 4.1.11. $Count^*$ *is not an ideal.*

Proof. We know that $\mathcal{M} \subseteq Count^*$ and $\mathcal{N} \subseteq Count^*$. But there exists $A \in \mathcal{N}$ such that $\mathbb{R} \setminus A \in \mathcal{M}$

□

4.2 Horbaczewska-Lindner results.

In [4] Seredyń ski posed several questions about * operation. For example he asked for which ideals $\mathcal{I}$ we have $\mathcal{I} = \mathcal{I}^{**}$. It turned out to be surprisingly difficult questions. The problem of the ideal of countable sets was solved by Solecki in [8] and then, by much easier methods, by Pawlikowski and Sabok in [9].

In 2018 Horbaczewska and Lindner published a paper [11] in which they presented a consisent answer for the ideals $\mathcal{M}$ and $\mathcal{N}$. Namely, they showed that under Continuum Hypothesis $\mathcal{M} = \mathcal{M}^{**}$ and $\mathcal{N} = \mathcal{N}^{**}$.

We will show that this result can be easily generalised.

The main ingredient of Horbaczewska and Lindner result is the following lemma.

Lemma 4.2.1. *(Horbaczewska, Lindner) For any $\mathcal{F} \subseteq \mathcal{P}(X)$ the following conditions are equivalent:*

- $\forall_{A \notin \mathcal{F}} (\mathcal{F} \cup \{A\})^* \neq \mathcal{F}^*$
- $\mathcal{F} = \mathcal{F}^{**}$

Proof. ($\Longrightarrow$) Assume $\forall_{A \notin \mathcal{F}} (\mathcal{F} \cup \{A\})^* \neq \mathcal{F}^*$. We have to show that $\mathcal{F}^{**} \subseteq \mathcal{F}$. If there exists $\mathcal{A} \in \mathcal{F}^{**} \setminus \mathcal{F}$, then $\mathcal{F} \cup \{A\} \subseteq \mathcal{F}^{**}$. So we have $\mathcal{F}^* = \mathcal{F}^{***} \subseteq (\mathcal{F} \cup \{A\})^*$. Obviously, $(\mathcal{F} \cup \{A\})^* \subseteq \mathcal{F}^*$. So we have contradiction.

($\Longleftarrow$) Assume $\mathcal{F} = \mathcal{F}^{**}$. Let $A \notin \mathcal{F}$. Then $\mathcal{F} \cup \{A\} \nsubseteq \mathcal{F}$. By the assumption we have $\mathcal{F} \cup \{A\} \nsubseteq \mathcal{F}^{**}$. Hence $\mathcal{F}^* = \mathcal{F}^{***} \nsubseteq (\mathcal{F} \cup \{A\})^*$. □

Theorem 4.2.2. *If $\mathcal{J} \subseteq \mathcal{P}(\mathbb{R})$ is a translation and reflection invariant proper σ-ideal with* $\mathrm{cof}(\mathcal{J}) \leqslant \mathfrak{c}$, $\mathrm{cov}(\mathcal{J}) = \mathfrak{c}$ *and $A \notin \mathcal{J}$, then*

$$(\mathcal{J} \cup \{A\})^* \neq \mathcal{J}$$

Proof. The proof repeats the proof in [11], but we don't need CH, only $\mathrm{cov}(\mathcal{J}) = \mathfrak{c}$.

Since $\mathrm{cof}(\mathcal{J}) \leqslant \mathfrak{c}$ there exists a family $\mathcal{F} \subseteq \mathcal{J}$ of subsets of X with $card\mathcal{F} = \mathfrak{c}$ such that for every set $I \in \mathcal{J}$ there exists a set $F \in \mathcal{F}$ covering I ($I \subseteq F$).

Let $\{z_\alpha\}_{\alpha<\mathfrak{c}}$ be an enumeration of $\mathbb{R}$ and let $\{F_\alpha\}_{\alpha<\mathfrak{c}}$ be an enumeration of all sets from $\mathcal{F}$. We build inductively sequences of reals $\{x_\alpha\}_{\alpha<\mathfrak{c}}$ and $\{r_\alpha\}_{\alpha<\mathfrak{c}}$. We start with two different numbers x_0 and r_0. Let $\lambda < \mathfrak{c}$. Suppose that we have already constructed $\{x_\alpha\}_{\alpha<\lambda}$ and $\{r_\alpha\}_{\alpha<\lambda}$ and we define x_λ and r_λ. Since $\bigcup_{\alpha_1,\alpha_2<\lambda}(F_{\alpha_1} + x_{\alpha_2}) \neq \mathcal{P}(\mathbb{R})$ we can choose $r_\lambda \notin \bigcup_{\alpha_1,\alpha_2<\lambda}(F_{\alpha_1} + x_{\alpha_2})$.

Let $B_\lambda = \mathbb{R} \backslash \bigcup_{\alpha_1,\alpha_2 \leqslant \lambda}(r_{\alpha_1} - F_{\alpha_2})$. Then $\mathbb{R} \backslash B_\lambda \in \mathcal{J}$. Obviously $\mathbb{R}\backslash(z_\lambda - B_\lambda) = z_\lambda - (\mathbb{R}\backslash B_\lambda) \in \mathcal{J}$. Since $A \notin \mathcal{J}$ then $A \nsubseteq \mathbb{R}\backslash(z_\lambda - B_\lambda)$, so $A \cap (z_\lambda - B_\lambda) \neq 0$. Hence there are $a_\lambda \in A$ and $b_\lambda \in B_\lambda$ such that $z_\lambda - b_\lambda = a_\lambda$. Let $x_\lambda = b_\lambda$. Using this procedure for every $\lambda < \mathfrak{c}$ we define $X = \{x_\alpha\}_{\alpha<\mathfrak{c}}$. Since $A + X = \mathbb{R}$, the set X does not belong to $(\mathcal{J} \cup \{A\})^*$. On the other hand, for $\alpha < \mathfrak{c}$ choosing $\lambda > \alpha$ we have $r_\lambda \notin X + F_\alpha$, so $X \in \mathcal{J}$.

□

Theorem 4.2.3. *If $\mathcal{J} \subseteq \mathcal{P}(\mathbb{R})$ is a translation and reflection proper σ-ideal with* $\mathrm{cof}(\mathcal{J}) \leqslant \mathfrak{c}$, $\mathrm{cov}(\mathcal{J}) = \mathfrak{c}$ *and $A \notin \mathcal{J}$, then*

$$\mathcal{J} = \mathcal{J}^{**}$$

Proof. From previous theorem we know that if $\mathcal{J} \subseteq \mathcal{P}(\mathbb{R})$ is a translation and reflection proper σ-ideal with $\mathrm{cof}(\mathcal{J}) \leqslant \mathfrak{c}$, $\mathrm{cov}(\mathcal{J}) = \mathfrak{c}$ and $A \notin \mathcal{J}$ then $(\mathcal{J} \cup \{A\})^* \neq \mathcal{J}$. So from Theorem 4.2.1 we have $\mathcal{J} = \mathcal{J}^{**}$.

□

From previous chapter we know that if we assume Martin's Axiom $\mathrm{cov}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \mathrm{cof}(\mathcal{M}) = \mathrm{cof}(\mathcal{N}) = \mathfrak{c}$.

Proposition 4.2.4. *Assuming Martin's Axiom we have:*

$$\mathcal{M} = \mathcal{M}^{**}$$

$$\mathcal{N} = \mathcal{N}^{**}$$

Proof. We know that $\mathcal{M}$ and $\mathcal{N}$ are $\sigma - ideals$ and assuming Martin's Axiom we have $\mathrm{cof}(\mathcal{M}) = \mathrm{cof}(\mathcal{N}) = \mathfrak{c}$ and $\mathrm{cov}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \mathfrak{c}$. So $\mathcal{M}$ and $\mathcal{N}$ satisfies assumptions of Theorem 4.2.3 $\square$

Proposition 4.2.5. *Assume Martin's Axiom (or just* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$*)*

$$\mathcal{M} = \mathcal{SMZ}^*$$

Proof. Assuming Marin's Axiom $\mathcal{M}$ satisfies assumptions of Theorem 4.2.3, so $\mathcal{M} = \mathcal{M}^{**}$ and since $\mathcal{M}^* = \mathcal{SMZ}$ we have $\mathcal{M} = \mathcal{SMZ}^*$ $\square$

Proposition 4.2.6. *Assume Martin's Axiom (or just* $\mathrm{cov}(\mathcal{N}) = \mathfrak{c}$*)*

$$\mathcal{N} = \mathcal{SM}^*$$

Proof. Assuming Marin's Axiom $\mathcal{N}$ satisfies assumptions of Theorem 4.2.3, so $\mathcal{N} = \mathcal{N}^{**}$ and since $\mathcal{N}^* = \mathcal{SM}$ we have $\mathcal{N} = \mathcal{SM}^*$ $\square$

The natural question is if the assertions of the above theorem hold true without assuming any additional axioms. The following results show that the answer is negative.

Theorem 4.2.7. *Borel Conjecture implies* $\mathcal{M} \neq \mathcal{M}^{**}$.

Proof. From Borel Conjecture we know that $\mathcal{SMZ} = Count$, so we have $\mathcal{M}^* = Count$. So $\mathcal{M}^{**} = Count^*$. With Proposition 4.1.8 we have $\mathcal{M} \neq Count^*$. So we have $\mathcal{M} \neq \mathcal{M}^{**}$.

$\square$

Theorem 4.2.8. *Dual Borel Conjecture implies* $\mathcal{N} \neq \mathcal{N}^{**}$.

Proof. From dual Borel Conjecture we know that $\mathcal{SM} = Count$, so we have $\mathcal{N}^* = Count$. So $\mathcal{N}^{**} = Count^*$ .With Proposition 4.1.8 we have $\mathcal{N} \neq Count^*$. So we have $\mathcal{N} \neq \mathcal{N}^{**}$. $\square$

So, there is a natural problem:

Problem 4.2.9. Is the statement $\mathcal{M} \neq \mathcal{M}^{**}$ equivalent with the Borel Conjecture? Is the statement $\mathcal{N} \neq \mathcal{N}^{**}$ equivalent with the dual Borel Conjecture?

We conjecture that the answers to the above problems are negative but we could not find any other reason for $\mathcal{M}$ not being $\mathcal{M}^{**}$ than Borel Conjecture (and similarly for $\mathcal{N}$ and dual Borel Conjecture).

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