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CAT(0)-boundaries of free products are dense amalgams

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Abstract:

Let $\Gamma = G * H$ be a free product of infinite groups acting geometrically on a proper CAT(0) space X. We show that the CAT(0)-boundary ∂X of X is homeomorphic to the dense amalgam of the limit sets of factors ΛG and ΛH.

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1 Introduction

One of the most important aspects of the geometric group theory are groups acting geometrically on the spaces of non-positive curvature, one of the most discussed types of which are CAT(0) spaces. It turns out that for a CAT(0) space X there can be defined the so called boundary at infinity ∂X of a space X. Since the boundary at infinity depends mostly on global properties of the given CAT(0) space, then a natural hypothesis can be formulated: for every group Γ acting geometrically on CAT(0) spaces X_1 and X_2, the boundaries ∂X_1 and ∂X_2 are homeomorphic. It turns out, however, that this is not true; the example of a group Γ that acts on two spaces X_1 and X_2 with non-homeomorphic boundaries was first given by C. B. Croke and B. Kleiner in [3]. Although the general result is false, we can still obtain some information about the boundary ∂X from the group Γ that acts geometrically on X.

In this paper I will prove that a boundary of a CAT(0) space X on which a free product of infinite groups $\Gamma = G * H$ acts geometrically can be described in terms of limit sets ΛG and ΛH of groups G and H. Formally, this result is stated in the following main theorem:

Main theorem

Let $\Gamma = G * H$ be a free product of infinite groups and suppose that Γ acts geometrically on a proper CAT(0) space X. Then ∂X can be expressed in terms of ΛG and ΛH in the following way:

$$\partial X \cong \widetilde{\sqcup}(\Lambda G, \Lambda H),$$

where $\widetilde{\sqcup}$ denotes the operation of dense amalgam of compact metric spaces.

Section 2 contains basic definitions and lemmas that will be used throughout the paper and are sufficiently well described in the literature. This section splits into four parts describing CAT(0) spaces and their boundaries, geometric group actions, free products of groups and dense amalgams. Most of the proofs in this chapter have been omitted and replaced only by relevant literature references. Section 3 consists of definitions and lemmas concerning limit sets that are also very important and used throughout this paper but were only vaguely mentioned in available sources. Section 4 introduces the concept of R-separation and proves important, technical lemmas that are the basis of reasoning in the next two chapters. Section 5 consists of six general but technical lemmas that are necessary for proving lemmas in the next section. Section 6 is focused on describing geodesic rays in $\partial_{x_0} X$. In the first part of this section the notion of generating sequences is described in such a way that it coincides with definition of prolongation given in Section 4 and definitions of limit sets from Section 3. In the second part of this section, several lemmas describing geodesic rays and estimating distances between them are proven. Finally the main theorem is proven in Section 7. Section 8 is written in a slightly looser language and describes open problems that were encountered during the writing of this paper and presents some ideas about how the main theorem could be possibly generalised.

2 Preliminaries

In this section I will present the most important facts that will be used throughout this paper. Those definitions and theorems are mostly well-known and were described in detail in [2], [1], [5] and [6]. For several of the lemmas, I found it appropriate to present their proofs, however in most cases I chose to refer the reader to the relevant sources.

2.1 CAT(0) spaces

Definition 2.1 (Geodesic metric space)
Let (X, d) be a metric space, let $x_1, x_2 \in X$ and $d(x_1, x_2) = D$. We will say that $\gamma : [0, D] \to X$ is a *geodesic path* between x_1 and x_2 if $\gamma(0) = x_1$, $\gamma(D) = x_2$ and for all $t_1, t_2 \in [0, D]$ we have $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$. We will say that the constant D is the *lenght* of the geodesic path γ.

We will say that a metric space (X, d) is *geodesic* if for each pair of points x_1, x_2 there is a geodesic path between x_1 and x_2.

Definition 2.2 (Geodesic triangle)
Let (X, d) be a geodesic space. An object Δ consisting of three points $A_1, A_2, A_3 \in X$ and three geodesic segments $\gamma_{[A_1,A_2]}, \gamma_{[A_1,A_3]}, \gamma_{[A_2,A_3]}$ between A_1 and A_2, A_1 and A_3, A_2 and A_3 respectively is called a *geodesic triangle.*

Definition 2.3 (Comparison triangle)
Let (X, d) be a geodesic metric space and let $A_1, A_2, A_3 \in X$. Any triple of points $A_1^*, A_2^*, A_3^* \in \mathbb{R}^2$ is called a *comparison triple* for A_1, A_2, A_3 if for any $i, j \in \{1, 2, 3\}$ the equality holds

$$d(A_i, A_j) = d_e(A_i^*, A_j^*),$$

where d_e is the euclidean metric on $\mathbb{R}^2$. Moreover we will denote the unique geodesic between A_i^*, A_j^* in $(\mathbb{R}^2, d_e)$ as $\eta_{[A_i^*,A_j^*]}$. A geodesic triangle consisting of points A_i^* and geodesic segments $\eta_{[A_i^*,A_j^*]}$ is called a *comparison triangle* for A_1, A_2, A_3.

Fact 2.4
For every triple of points A_1, A_2, A_3 from geodesic space X there exists a comparison triangle for A_1, A_2, A_3, and it is unique up to congruence.

Definition 2.5
Let (X, d) be a geodesic space, $A, B, C \in X$, and let $\gamma_{[A,B]}$, $\gamma_{[A,C]}$, $\gamma_{[B,C]}$ be geodesic paths between A and B, A and C, B and C respectively. We will say that a geodesic triangle consisting of points A, B, C and geodesics $\gamma_{[A,B]}, \gamma_{[A,C]}, \gamma_{[B,C]}$ in X is *thin* if for any comparison triangle consisting of points $A^*, B^*, C^* \in \mathbb{R}^2$ and geodesic segments $\eta_{[A^*,B^*]}, \eta_{[A^*,C^*]}, \eta_{[B^*,C^*]}$ the following conditions holds:

$$\forall_{s\in[0,d(A,B)]}\forall_{t\in[0,d(A,C)]}\ d(\gamma_{[A,B]}(s), \gamma_{[A,C]}(t)) \leqslant d_e(\eta_{[A^*,B^*]}(s), \eta_{[A^*,C^*]}(t)),$$

$$\forall_{s\in[0,d(A,B)]}\forall_{t\in[0,d(B,C)]}\ d(\gamma_{[A,B]}(s), \gamma_{[B,C]}(t)) \leqslant d_e(\eta_{[A^*,B^*]}(s), \eta_{[B^*,C^*]}(t)),$$

$$\forall_{s\in[0,d(A,C)]}\forall_{t\in[0,d(B,C)]}\ d(\gamma_{[A,C]}(s), \gamma_{[B,C]}(t)) \leqslant d_e(\eta_{[A^*,C^*]}(s), \eta_{[B^*,C^*]}(t)).$$

Definition 2.6 (CAT(0) space)
A space (X, d) is called a *CAT(0) space* if it is geodesic and if every geodesic triangle in X is thin.

Definition 2.7 (Proper metric space)
We will say that a space (X, d) is *proper* if every closed ball in X is compact.

Fact 2.8
Every proper metric space is complete.

Definition 2.9
Let (X, d) be a CAT(0) space and let γ_1 and γ_2 be geodesic paths of equal length D in a CAT(0) space (X, d). Then we define the *distance function* $f : [0, D] \to [0, \infty)$ between γ_1, γ_2 as follows:

$$f(t) = d(\gamma_1(t), \gamma_2(t)).$$

The following lemma is well-known and considered an important result about geometry of CAT(0) spaces. A detailed proof can be found in [2] (Proposition II.2.2).

Lemma 2.10
Let γ_1 and γ_2 be geodesic paths of equal length D in a CAT(0) space (X, d). Then the distance function between γ_1 and γ_2 is convex.

Definition 2.11 (Geodesic ray)
Given a CAT(0) space (X, d), we say that a curve $\gamma : [0, \infty) \to X$ is a *geodesic ray* if for every $s, t \geqslant 0$ the following equality holds:

$$d(\gamma(s), \gamma(t)) = |s - t| \, .$$

Moreover we will say that γ starts at x_0 if $\gamma(0) = x_0$

Definition 2.12 (Asymptotic geodesic rays)
Two geodesic rays γ_1 and γ_2 are *asymptotic* if there exists a constant $K \geqslant 0$ such that for every $t \geqslant 0$ we have:

$$d(\gamma_1(t), \gamma_2(t)) \leqslant K.$$

We denote asymptoticity of γ_1 and γ_2 by $\gamma_1 \sim \gamma_2$.

Fact 2.13
The relation $\sim$ on the set of geodesic rays in a CAT(0) space is an equivalence relation.

The following lemma is important for defining the boundary at infinity of a CAT(0) space. For a proof see [2] (Proposition II.8.2).

Lemma 2.14
Let (X, d) be a proper CAT(0) space. For every point $x_0 \in X$ and for every geodesic ray γ there exist a unique geodesic ray γ' starting at x_0 such that $\gamma \sim \gamma'$.

Definition 2.15
Let (X, d) be a proper CAT(0) space. We will denote the set of all geodesic rays starting at x_0 by $\partial_{x_0} X$.

Definition 2.16 (Boundary at infinity)
Let (X, d) be a proper CAT(0) space. The set ∂X is defined as:

$$\partial X = \{\gamma : \gamma \text{ is a geodesic ray in } X\} / \sim .$$

∂X is sometimes called the *boundary at infinity* of X (or simply *boundary* of X).

Fact 2.17
There is a canonical bijection between $\partial_{x_0} X$ and ∂X that maps elements of $\partial_{x_0} X$ to their equivalence classes in ∂X.

Definition 2.18 (Topology on the boundary)
There is a natural topology τ_{x_0} on $\partial_{x_0}X$ generated by the basis of open sets

$$N_{x_0}(R, \varepsilon, \gamma) = \{\gamma' \in \partial_{x_0}X :\ d(\gamma(R), \gamma'(R)) < \varepsilon\},$$

where $R > 0, \varepsilon > 0$ and γ is a geodesic ray starting at x_0.

We define a topology on ∂X as the topology induced from τ_{x_0} by the canonical bijection between $\partial_{x_0}X$ and ∂X.

Fact 2.19
The topology on ∂X does not depend on the choice of the base point x_0.

The following lemma is a particularly useful fact about boundaries at infinity. For more details see [2] (Definition II.8.6).

Lemma 2.20
For a proper CAT(0) space X, its boundary at infinity ∂X is compact.

It turns out that boundaries of proper CAT(0) spaces are metrizable. This fact can be most directly deduced from the following observation from [5] (Proposition 9.6).

Definition 2.21 (Osajda's metric)
Let (X, d) be a proper CAT(0) space. For a positive constant A we define *Osajda's metric d_A* on $\partial_{x_0}X$ as follows:

$$d_A(\gamma, \gamma') = \begin{cases} 0 \text{ iff } \gamma = \gamma' \\ \frac{1}{M}, \text{ where } M = \inf\{t \in [0, \infty) : d(\gamma(t), \gamma'(t)) \geqslant A\} & \text{otherwise.} \end{cases}$$

Lemma 2.22
d_A is a metric on $\partial_{x_0}X$ and it is compatible with the topology τ_{x_0} on $\partial_{x_0}X$.

2.2 Švarc-Milnor lemma

The concepts below are explained in more details in section I.8 of [2].

Definition 2.23 (Geometric group action)
Let (X, d) be a proper geodesic space and let Γ be a group that acts on X by isometries. Then we say that the action of Γ is

- *properly discontinuous* if for every compact set $K \subseteq X$ the set $\{g \in \Gamma : K \cap g \cdot K\}$ is finite;
- *cocompact* if there exist a compact set $K \subseteq X$ such that $\bigcup_{g\in\Gamma} g \cdot K = X$.

We will say that action of Γ is *geometric* if it is properly discontinuous and cocompact. We will denote the fact that Γ acts on X geometrically by $\Gamma \curvearrowright X$.

Definition 2.24 (Word metric)
Let Γ be any group and $\mathcal{A}$ its set of generators. Then we define the *word metric* associated with $\mathcal{A}$ as a metric on Γ given by the formula $d_{\mathcal{A}}(g, g') = n$, where n is the length of $g'g^{-1}$ expressed as a shortest word over generators from $\mathcal{A}$ and their inverses. Moreover we will denote $|g|_{\mathcal{A}} = d_{\mathcal{A}}(1, g)$.

Definition 2.25 (Quasi-isometries)
Let (Y, d_Y) and (X, d_X) be two metric spaces. Given some constants $L \geqslant 1$ and $A \geqslant 0$, a map $f : Y \to X$ is called an (L, A)-*quasi-isometric embedding* if for every $y_1, y_2 \in Y$ we have the following estimates:

$$\frac{1}{L} d_Y(y_1, y_2) - A \leqslant d_X(f(y_1), f(y_2)) \leqslant L d_Y(y_1, y_2) + A.$$

A map $f : Y \to X$ is A-*quasi-dense* if for every $x \in X$ there is a $y \in Y$ such that $d_X(x, f(y)) \leqslant A$.

A map $f : Y \to X$ is called an (L, A)-*quasi-isometry* if f is both an (L, A)-quasi-isometric embedding and is A-quasi-dense. A map is called a *quasi-isometry* if it is an (L, A)-quasi-isometry for some $L \geqslant 1, A \geqslant 0$.

The following theorem is one of the most important facts about groups acting geometrically on geodesic spaces. It was presented and proven in [2] (Proposition I.8.19).

Lemma 2.26 (Švarc-Milnor lemma)
Let Γ be a group acting geometrically on a geodesic space (X, d). Then Γ is finitely generated and for any finite generating set $\mathcal{S}$ and any $x_0 \in X$ the map $f : \Gamma \to X$ given by the formula $f(g) = g \cdot x_0$ is a quasi-isometry between $(\Gamma, d_{\mathcal{S}})$ and (X, d).

2.3 Free products

Definitions and lemmas in this subsection are described in more detail and proven in section IV of [1].

Definition 2.27 (Free product)
Let G and H be groups with presentations $\langle S_G; R_G \rangle$ and $\langle S_H; R_H \rangle$ respectively. Then the *free product* of G and H is defined as the group $G * H = \langle S_G \sqcup S_H; R_G \sqcup R_H \rangle$, where $\sqcup$ denotes the operation of disjoint union.

Lemma 2.28
The free product $G * H$ is well-defined and does not depend on choice of presentations for G and H.

Lemma 2.29 (Normal form lemma)
Each element w from the group $G*H$ can be uniquely written in the form $w = g_1 h_1 ... g_n h_n$, where $n \geqslant 1$, $g_1 \in G$, $g_2, ..., g_n \in G \setminus \{1\}$, $h_1, h_2, ..., h_{n-1} \in H \setminus \{1\}$ and $h_n \in H$. We will call this form the normal form of w.

Fact 2.30
Let G and H be groups with presentations $\langle S_G; R_G \rangle$ and $\langle S_H; R_H \rangle$ respectively. Moreover let $w \in G * H$ and let $g_1 h_1 ... g_N h_N$ be the normal form of w. Then

$$|g_1 h_1 ... g_N h_N|_{\mathcal{A}} = |g_1|_{\mathcal{A}} + |h_1|_{\mathcal{A}} + ... + |g_N|_{\mathcal{A}} + |h_N|_{\mathcal{A}},$$

where $\mathcal{A} = S_G \sqcup S_H$.

2.4 Dense amalgams

Definition 2.31
Let X be a metrisable topological space and let $\mathcal{Y} = \{Y_1, Y_2, ...\}$ be a countable family of

subsets of X. We say that $\mathcal{Y}$ is *null with respect to the metric* d on X if $\lim_{n\to\infty} \operatorname{diam}(Y_n) = 0$ where $\operatorname{diam}(A)$ is the diameter of set A in the metric d.

Lemma 2.32
Let X be a compact topological space and let d_1, d_2 be metrics compatible with the topology on X. We will denote diameters in metrics d_1, d_2 as $\operatorname{diam}_1, \operatorname{diam}_2$ respectively. Moreover let Y_n be a family of subsets of X such that $\lim_{n\to\infty} \operatorname{diam}_1(Y_n) = 0$. Then we have $\lim_{n\to\infty} \operatorname{diam}_2(Y_n) = 0$.

Proof. Let us assume on the contrary that the diameters $\operatorname{diam}_2(Y_n)$ do not converge to 0. Then there exist $\varepsilon > 0$ such that there exist a subsequence Y_{n_k} such that $\operatorname{diam}_2(Y_{n_k}) > \varepsilon$. Let $y_{n_k}, y'_{n_k} \in Y_{n_k}$ be such that $d_2(y_{n_k}, y'_{n_k}) \geqslant \varepsilon$. Now we take such a subsequence n_{k_l} of n_k that $\lim_{l\to\infty} y_{n_{k_l}}$ and $\lim_{l\to\infty} y'_{n_{k_l}}$ both exist. Since $\lim_{l\to\infty} \operatorname{diam}_1(Y_{n_{k_l}}) = 0$, then $\lim_{l\to\infty} y_{n_{k_l}} = \lim_{l\to\infty} y'_{n_{k_l}}$. Therefore $\lim_{l\to\infty} d_2(y_{n_{k_l}}, y_{n_{k_l}}) = 0$ and thus $d_2(y_{n_{k_l}}, y_{n_{k_l}}) < \varepsilon$ for sufficiently large l. This contradiction completes the proof. □

Definition 2.33 (Null family)
We say that the family $\mathcal{Y}$ is *null* if it is null with respect to any metric compatible with the topology on X.

An important concept apperaring in the main theorem of this paper is the operation of dense amalgam. This operation was described in great detail in [6].

Definition 2.34 (Dense amalgam of compact metric spaces)
Let $X_1, X_2, ..., X_n$ be a collection of nonempty compact metric spaces. Then the dense amalgam of $X_1, X_2, ..., X_n$ is defined as the unique (up to homeomorphism) compact metric space Y that can be equipped with a countable infinite family $\mathcal{Y}$ of subsets of Y partitioned as $\mathcal{Y} = \mathcal{Y}_1 \sqcup \mathcal{Y}_2 \sqcup ... \sqcup \mathcal{Y}_n$ such that

(i) The subsets in $\mathcal{Y}$ are pairwise disjoint and for each $i \in \{1, 2, ..., n\}$ the family $\mathcal{Y}_i$ consist of embedded copies of the space X_i;

(ii) the family $\mathcal{Y}$ is null;

(iii) each $Z \in \mathcal{Y}$ is a boundary subset of Y (i.e. its complement is dense);

(iv) for each i, the union of the family $\mathcal{Y}_i$ is dense in Y;

(v) any two points of Y which do not belong to the same subset from $\mathcal{Y}$ can be separated from each other by a clopen subset $Q \subseteq Y$ which is $\mathcal{Y}$-saturated (i.e. such that any element of $\mathcal{Y}$ is either contained in or disjoint with Q).

The dense amalgam of nonempty, compact metric spaces $X_1, ..., X_n$ will be denoted by $\widetilde{\sqcup}(X_1, ..., X_n)$.

3 Limit sets

Definition 3.1
Let (X, d) be a CAT(0) space and let γ_n be a sequence consisting of geodesic rays or geodesic paths. Moreover let $\gamma : [0, \infty) \to X$ be a geodesic ray. Then we will write

$$\lim_{n\to\infty} \gamma_n = \gamma$$

if for every $t \geqslant 0$ the point $\gamma_n(t)$ is well-defined for almost every $n \in \mathbb{N}$ and we have $\lim_{n\to\infty} \gamma_n(t) = \gamma(t)$.

Lemma 3.2
Let γ_n be a sequence of geodesic paths of length d_n in a proper CAT(0) space (X, d) starting at a base point $x_0 \in X$. If $\lim_{n\to\infty} d_n = \infty$, then there exist a geodesic ray γ and a subsequence γ_{n_k} of γ_n such that $\lim_{k\to\infty} \gamma_{n_k} = \gamma$.

Proof. Since $\lim_{n\to\infty} d_n = \infty$, for each $R > 0$ the sequence $\gamma_n(R)$ is well-defined for almost every $n \in \mathbb{N}$. Moreover, $\gamma_n(R) \in \overline{B}(x_0, R)$ for each n such that $d_n \geqslant R$, so since X is proper then $\overline{B}(x_0, R)$ is compact, so there exist a subsequence $\gamma_{n_k}(R)$ of $\gamma_n(R)$ that converges. From Lemma 2.10, we can conclude that for all $t \in [0, R]$ we have the following estimation:

$$d(\gamma_{n_k}(t), \gamma_{n_{k'}}(t)) \leqslant \frac{t}{R} d(\gamma_{n_k}(R), \gamma_{n_{k'}}(R)) \leqslant d(\gamma_{n_k}(R), \gamma_{n_{k'}}(R)).$$

So, since $\gamma_{n_k}(R)$ was a Cauchy sequence, then $\gamma_{n_k}(t)$ is also a Cauchy sequence, so it converges. Now we will define the sequences $\gamma_n^{(k)}$. Let $\gamma_n^{(0)} = \gamma_n$ and $\gamma_n^{(k+1)}$ will be such a subsequence of $\gamma_n^{(k)}$ that $\gamma_n^{(k+1)}(k+1)$ converges. We will denote $\widetilde{\gamma}_k = \gamma_k^{(k)}$. Obviously for all $k \in \mathbb{N}$ the sequence $\widetilde{\gamma}_n(k)$ from some point is a subsequence of $\gamma_n^{(k)}(k)$. Therefore from the reasoning above we know that $\widetilde{\gamma}_n(t)$ is convergent for any $t \in [0, \infty)$. Now let $\gamma(t) = \lim_{n\to\infty} \widetilde{\gamma}_n(t)$. We will show that γ is a geodesic ray. Let $s, t \in [0, \infty)$. From the triangle inequality we have:

$$\begin{aligned} d(\gamma(t), \gamma(s)) &\leqslant d(\gamma(t), \widetilde{\gamma}_n(t)) + d(\widetilde{\gamma}_n(t), \widetilde{\gamma}_n(s)) + d(\widetilde{\gamma}_n(s), \gamma(s)) \\ &= |s - t| + d(\gamma(t), \widetilde{\gamma}_n(t)) + d(\gamma(s), \widetilde{\gamma}_n(s)) \\ d(\gamma(t), \gamma(s)) &\geqslant d(\widetilde{\gamma}_n(t), \widetilde{\gamma}_n(s)) - d(\widetilde{\gamma}_n(s), \gamma(s)) - d(\widetilde{\gamma}_n(s), \gamma(s)) \\ &= |s - t| - d(\gamma(t), \widetilde{\gamma}_n(t)) - d(\gamma(s), \widetilde{\gamma}_n(s)) \end{aligned}$$

Therefore, from the squeeze theorem, $d(\gamma(t), \gamma(s)) = |t - s|$, which shows that γ is a geodesic ray and ends the proof. □

The following observation concerning CAT(0) spaces will be needed in the later parts of this section.

Lemma 3.3 (Triangle cutting lemma)
Let A, B, C be any points in a CAT(0) space (X, d), such that $d(A, B) = c$, $d(A, C) = b$ and $d(B, C) = a$ where $b \geqslant c$. Then $d(B, \gamma_{[A,C]}(c)) \leqslant a$.

Proof. Let A^*, B^*, C^* be the vertices of a comparison triangle for A, B, C and let D^* be such a point on the edge A^*C^* that $|A^*D^*| = |A^*B^*|$. Since $A^*B^*D^*$ is an isosceles triangle then $\sphericalangle A^*D^*B^* \leqslant \frac{\pi}{2}$ and thus $\sphericalangle C^*D^*B^* \geqslant \frac{\pi}{2}$. Therefore from the definition of CAT(0) space and the cosine formula we know that

$$\begin{aligned} d(B, C) = |B^*C^*| &= \sqrt{|B^*D^*|^2 + |D^*C^*|^2 - 2|B^*D^*||D^*C^*| \cos(\sphericalangle C^*D^*B^*)} \\ &\geqslant \sqrt{|B^*D^*|^2} = |B^*D^*| \geqslant d(B, \gamma_{[A,C]}(c)). \end{aligned}$$

□

Limit sets are subsets of boundary that are necessary to formulate the main theorem of this paper. They were described in [4] (section 3).

Definition 3.4
Let X be a proper CAT(0) space and let $A \subseteq X$. Then the *limit set of A with respect to x_0* is defined as

$$\Lambda_{x_0} A = \left\{ \gamma \in \partial_{x_0} X : \gamma = \lim_{n\to\infty} \gamma_{[x_0,a_n]} \text{ for } a_n \in A \right\},$$

where $x_0 \in X$ is a base point.

The lemma below guarantees that the limit sets are well-defined and do not depend on the choice of a base point x_0.

Lemma 3.5
Let (X,d) be a proper CAT(0) space, let $A \subseteq X$ and let $x_0, x_1 \in X$. Then for every $\gamma \in \Lambda_{x_0} A$ there exist $\gamma' \in \Lambda_{x_1} A$ such that $\gamma \sim \gamma'$.

Proof. Let $a_n \in A$ be such sequence that $\gamma = \lim_{n\to\infty} \gamma_{[x_0,a_n]}$. Since

$$\lim_{n\to\infty} d(x_1, a_n) \geqslant \lim_{n\to\infty} d(x_0, a_n) - d(x_0, x_1) = \infty$$

then from Lemma 3.2 there is such subsequence a_{n_k} of a_n that there exist a geodesic ray $\gamma' = \lim_{k\to\infty} \gamma_{[x_1,a_{n_k}]} \in \Lambda_{x_1} A$. We will show that $\gamma \sim \gamma'$. Let $\varepsilon > 0, t \geqslant 0$ be any numbers and let $k \in \mathbb{N}$ be such that $d(\gamma(t), \gamma_{[x_0,a_{n_k}]}(t)) < \varepsilon$ and $d(\gamma'(t), \gamma_{[x_1,a_{n_k}]}(t)) < \varepsilon$. Without loss of generality we assume that $d(x_0, a_{n_k}) \geqslant d(x_1, a_{n_k})$, let $x_2 = \gamma_{[x_0,a_{n_k}]}(d(x_1, a_{n_k}))$. From triangle inequality we have $d(x_0, a_{n_k}) - d(x_1, a_{n_k}) \leqslant d(x_0, x_1)$ and therefore from convexity of the distance function for geodesic segments $\gamma_{[x_1,a_{n_k}]}$, γ_{x_0,x_2} we know that

$$d(\gamma_{[x_0,a_{n_k}]}(t), \gamma_{[x_1,a_{n_k}]}(t)) = d(\gamma_{[x_0,x_2]}(t), \gamma_{[x_1,a_{n_k}]}(t)) \leqslant \max\{d(x_0,x_1), d(a_{n_k}, x_2)\} = d(x_0,x_1).$$

Therefore

$$\begin{aligned} d(\gamma(t), \gamma'(t)) &\leqslant d(\gamma(t), \gamma_{[x_0,a_{n_k}]}(t)) + d(\gamma_{[x_0,a_{n_k}]}(t), \gamma_{[x_1,a_{n_k}]}(t)) \\ &\quad + d(\gamma_{[x_1,a_{n_k}]}(t), \gamma'(t)) \leqslant d(x_0,x_1) + 2\varepsilon, \end{aligned}$$

which proves that $\gamma \sim \gamma'$. □

Definition 3.6 (Limit set)
The limit set of A is defined as

$$\Lambda A = \left(\bigcup_{x_0 \in X} \Lambda_{x_0} A \right) / \sim,$$

where $\sim$ is the relation of asymptoticity on geodesic rays. From Lemma 3.5 we conclude that ΛA is naturally identified with set $\Lambda_{x_0} A$ for any base point $x_0 \in X$. Topology on the set ΛA is defined as the topology induced from ∂X.

Moreover, let Γ be a group such that $\Gamma \curvearrowright X$, let $G < \Gamma$ be a subgroup of Γ and let $k \in \Gamma$ be an element of Γ. Then we can define the limit set of a subgroup and limit set of a coset as follows:

$$\Lambda G = \Lambda(G \cdot x)$$
$$\Lambda(kG) = \Lambda(kG \cdot x)$$

for an $x \in X$.

Lemma 3.7
Let Γ be any group such that $\Gamma \curvearrowright X$ for a proper CAT(0) space (X,d), let $G < \Gamma$ and let $x, x' \in X$. Then $\Lambda(G \cdot x) = \Lambda(G \cdot x')$.

Proof. Let $x_0 \in X$ be a fixed point and let $g_n \in G$ be such that $\lim_{n\to\infty} \gamma_{[x_0, g_n \cdot x]} = \gamma$ for a geodesic ray γ. Let $t \geqslant 0, \varepsilon > 0$ be any numbers and let $n \in \mathbb{N}$ be such that $d(x_0, g_n \cdot x) \geqslant \max\left\{\frac{2td(x,x')}{\varepsilon} + d(x,x'), t + d(x,x')\right\}$ and $d(\gamma_{[x_0, g_n\cdot x]}(t), \gamma(t)) < \frac{\varepsilon}{2}$. Then we have

$$d(x_0, g_n \cdot x') \geqslant d(x_0, g_n \cdot x) - d(g_n \cdot x, g_n \cdot x') = d(x_0, g_n \cdot x) - d(x,x') \geqslant \max\left\{\frac{2td(x,x')}{\varepsilon}, t\right\}.$$

Let $s = \min\{d(x_0, g_n \cdot x), d(x_0, g_n \cdot x')\}$. From triangle cutting lemma and convexity of the distance function we can now estimate

$$d(\gamma(t), \gamma_{[x_0, g_n\cdot x']}(t)) \leqslant d(\gamma(t), \gamma_{[x_0, g_n\cdot x]}(t)) + d(\gamma_{[x_0, g_n\cdot x]}(t), \gamma_{[x_0, g_n\cdot x']}(t))$$

$$< \frac{\varepsilon}{2} + \frac{t}{s} d(\gamma_{[x_0, g_n\cdot x]}(s), \gamma_{[x_0, g_n\cdot x']}(s)) \leqslant \frac{\varepsilon}{2} + \frac{t}{s} d(g_n \cdot x, g_n \cdot x') = \frac{\varepsilon}{2} + \frac{t}{s} d(x,x') = \varepsilon,$$

which ends the proof. □

Lemma 3.8
Let X be a proper CAT(0) space, let $x \in X$ and let Γ a group be such that $\Gamma \curvearrowright X$. Moreover let $G < \Gamma$. Then for any two cosets kG and $k'G$ we have $\Lambda kG \cong \Lambda k'G$.

Proof. We will show that for any $k \in \Gamma$ we have $\Lambda G \cong \Lambda kG$. Let $x \in X$ be any fixed point. Then we have

$$\Lambda(kG) = \Lambda(kG \cdot x) \cong \Lambda_x(kG \cdot x) \cong \Lambda_{k\cdot x}(kG \cdot x) \cong \Lambda_x(G \cdot x) \cong \Lambda(G \cdot x) = \Lambda G,$$

which ends the proof. □

4 Separation lemma

Definition 4.1 (Separating set)
Let (X,d) be a pathwise connected space and let $x_1, x_2 \in X$. We will say that a set $K \subseteq X$ *separates* x_1 from x_2 if x_1 and x_2 are in different pathwise connected components of $X \setminus K$.

Definition 4.2 (R-separating set)
Let (X,d) be a metric space and $x_1, x_2 \in X$. We will say that a set $I \subseteq X$ *R-separates* x_1 from x_2 if there exist $S_1, S_2 \subseteq X$ such that $X \setminus I = S_1 \sqcup S_2$, $x_1 \in S_1$, $x_2 \in S_2$, $d(x_1, I) \geqslant R$, $d(x_2, I) \geqslant R$, and $d(S_1, S_2) \geqslant R$.

Definition 4.3 (A-neighbourhoods)
Let (X,d) be a metric space and $K \subseteq X$. We define the *A-neighbourhood* of K in X as the set

$$\{x \in X : d(x,K) \leqslant A\}.$$

We will denote the A-neighbourhood of a set K by $N_A(K)$.

Lemma 4.4 (R-separation lemma)
Let (Y, d_Y) be a metric space, let (X, d_X) be a pathwise connected space and let $f : Y \to X$ be a (L, A)-quasi-isometry. Moreover let y_1, y_2 be elements of Y such that there exists a set $I \subseteq Y$ $(3LA + \varepsilon)$-separates y_1 from y_2 for some $\varepsilon > 0$. Then the set $K = N_A(f(I))$ separates $f(y_1)$ from $f(y_2)$ in X.

Proof. Firstly, we acknowledge the fact that $f(y_1), f(y_2) \notin K$. Without loss of generality we will show that $f(y_1) \notin K$. Assume on the contrary that $f(y_1) \in K$. Then $d_X(f(I), f(y_1)) \leqslant A$, so by the definition of quasi-isometry we have the following estimate:

$$A \geqslant d_X(f(I), f(y_1)) = \inf_{y \in I} d_X(f(y), f(y_1)) \geqslant \inf_{y \in I} \frac{1}{L} d_Y(y, y_1) - A$$

$$\geqslant \frac{3LA + \varepsilon}{L} - A = 2A + \frac{\varepsilon}{L} > A.$$

We get a contradiction that ends this part of the proof.

Now let us assume that K does not separate $f(y_1)$ from $f(y_2)$. Then there exists a path c in $X \setminus K$ between $f(y_1)$ and $f(y_2)$, so there is also a sequence $x_1, x_2, ..., x_n$ such that $x_1 = f(y_1), x_n = f(y_2)$ and for all $i \in \{1, 2, ..., n-1\}$ we have $d_X(x_i, x_{i+1}) < \frac{\varepsilon}{L}$. From the definition of a quasi-isometry we know that there exists $y \in Y$ such that $d_X(x, f(y)) \leqslant A$, so let $z_1, z_2, ..., z_n \in Y$ be such that $d_X(x_i, f(z_i)) \leqslant A$ for all $i \in \{1, 2, ..., n\}$. But then for all $i \in \{1, 2, ..., n-1\}$ we have the following estimates:

$$d_Y(z_i, z_{i+1}) \leqslant L d_X(f(z_i), f(z_{i+1})) + LA \leqslant L(d_X(f(z_i), x_i) + d_X(x_i, x_{i+1}) + d_X(x_{i+1}, f(z_{i+1})))$$

$$+LA \leqslant L(A + \frac{\varepsilon}{L} + A) + LA < 3LA + \varepsilon.$$

Since the set I $(3LA+\varepsilon)$-separates z_1 from z_n, there exist sets S_1, S_2 such that we have $S_1 \sqcup S_2 = X \setminus I$, $z_1 \in S_1$, $z_2 \in S_2$, $d(x_1, I) \geqslant R$, $d(x_2, I) \geqslant R$ and $d(S_1, S_2)$. Therefore $z_1 \in S_1$ and $z_n \notin S_n$, so let $i_0 \in \{1, 2, ..., n\}$ be the biggest index such that $z_{i_0} \in S_1$. Then we have $z_{i_0+1} \notin S_2$: assume on the contrary that $z_{i_0+1} \in S_2$. We can now estimate that

$$3LA + \varepsilon \leqslant d_Y(S_1, S_2) \leqslant d_Y(z_{i_0}, z_{i_0+1}) < 3LA + \varepsilon.$$

This contradiction proves that $z_{i_0+1} \notin S_2$, therefore $z_{i_0+1} \in I$. But then we have the following estimate:

$$d_X(x_{i_0+1}, f(I)) \leqslant d_X(x_{i_0+1}, f(z_{i_0+1})) \leqslant A,$$

So $x_{i_0+1} \in K$ and we have a contradiction with our choice of $x_1, ..., x_n$, which ends the proof. □

Definition 4.5
Let $\Gamma = G * H$ be a free product of groups and let w, v be elements of Γ. Moreover let $w = g_1 h_1 ... g_n h_n$ and $v = \widetilde{g}_1 \widetilde{h}_1 ... \widetilde{g}_{\widetilde{n}} \widetilde{h}_{\widetilde{n}}$ be the normal forms of w and v. Then w is called a *prolongation* of v if

- $w \neq 1$ when $v = 1$;
- $n \geqslant \widetilde{n}$ and for all $i \in \{1, ..., \widetilde{n} - 1\}$ we have the equalities $\widetilde{g}_i = g_i$, $\widetilde{h}_i = h_i$, $\widetilde{g}_{\widetilde{n}} = g_{\widetilde{n}}$, and $h_{\widetilde{n}} \neq 1$ when $\widetilde{h}_{\widetilde{n}} = 1$;
- $n > \widetilde{n}$ and for all $i \in \{1, ..., \widetilde{n}\}$ we have the equalities $\widetilde{g}_i = g_i$, $\widetilde{h}_i = h_i$ otherwise.

Lemma 4.6
Let G and H be groups with presentations $\langle \mathcal{A}_G; R_G\rangle$ and $\langle \mathcal{A}_H; R_H\rangle$ respectively. Moreover, let $\Gamma = G * H$ and let $\mathcal{A} = \mathcal{A}_G \sqcup \mathcal{A}_H$ be a set of generators of Γ. Furthermore, let $R > 0$ and $g, g_1, g_2 \in \Gamma$. If g_1 is a prolongation of g, g_2 is not a prolongation of g, $d_{\mathcal{A}}(g, g_1) \geqslant 2R$ and $d_{\mathcal{A}}(g, g_2) \geqslant 2R$ then the set $I = B(g, R)$ R-separates g_1 from g_2 in $(\Gamma, d_{\mathcal{A}})$.

Proof. Let

$$S_1 = \{g' \in \Gamma \setminus I : g' \text{ is a prolongation of } g\}$$

and

$$S_2 = \{g' \in \Gamma \setminus I : g' \text{ is not a prolongation of } g\}$$

be sets from the definition of R-separation. Obviously $S_1 \sqcup S_2 = \Gamma \setminus I$, and moreover we have the estimates:

$$d_{\mathcal{A}}(g_1, I) \geqslant d_{\mathcal{A}}(g_1, g) - \sup_{g''\in I} d_{\mathcal{A}}(g, g'') > 2R - R = R$$

$$d_{\mathcal{A}}(g_2, I) \geqslant d_{\mathcal{A}}(g_2, g) - \sup_{g''\in I} d_{\mathcal{A}}(g, g'') > 2R - R = R,$$

hence we only need to show that $d_{\mathcal{A}}(S_1, S_2) \geqslant R$. Let $g' \in S_1, g'' \in S_2$ be any elements of S_1 and S_2 respectively. Without loss of generality let's assume that $g^{(1)}h^{(1)}...g^{(N)}h^{(N)}$ is the normal form of g and $h^{(N)} \neq 1$. Since g' is a prolongation of g then g' has a normal form $g^{(1)}h^{(1)}...g^{(N)}h^{(N)}g^{(N+1)}h^{(N+1)}...g^{(M)}h^{(M)}$. Moreover, since g'' is not a prolongation of g, g'' has the normal form $g^{(1)}h^{(1)}...g^{(L)}h^{(L)}\hat{g}^{(L+1)}\hat{h}^{(L+1)}...\hat{g}^{(K)}\hat{h}^{(K)}$ where $L < N$ and at least one of the inequalities $g^{(L+1)} \neq \hat{g}^{(L+1)}$, $h^{(L+1)} \neq \hat{h}^{(L+1)}$ hold. Without loss of generality let's assume that $g^{(L+1)} \neq \hat{g}^{(L+1)}$. We know that $d_{\mathcal{A}}(g', g'')$ is equal to the length of

$$\left(\hat{g}^{(K)}\right)^{(-1)} ... \left(\hat{h}^{(L+1)}\right)^{(-1)} \left(\hat{g}^{(L+1)}\right)^{(-1)} g^{(L+1)}h^{(L+1)}...g^{(N)}h^{(N)}g^{(N+1)}h^{(N+1)}...h^{(M)}$$

expressed as the shortest word over generators $\mathcal{A}$. Then

$$\left|\left(\hat{g}^{(K)}\right)^{(-1)} ... \left(\hat{h}^{(L+1)}\right)^{(-1)} \left(\hat{g}^{(L+1)}\right)^{(-1)} g^{(L+1)}h^{(L+1)}...g^{(N)}h^{(N)}g^{(N+1)}h^{(N+1)}...h^{(M)}\right|_{\mathcal{A}}$$

$$= \left|\left(\hat{g}^{(K)}\right)^{(-1)}\right|_{\mathcal{A}} + ... + \left|\left(\hat{g}^{(L+1)}\right)^{(-1)}\right|_{\mathcal{A}} + \left|g^{(L+1)}\right|_{\mathcal{A}} + ... + \left|h^{(N)}\right|_{\mathcal{A}} + \left|g^{(N+1)}\right|_{\mathcal{A}} + ... + \left|h^{(M)}\right|_{\mathcal{A}}$$

$$= \left|\left(\hat{g}^{(K)}\right)^{(-1)} ... \left(\hat{g}^{(L+1)}\right)^{(-1)} g^{(L+1)}...h^{(N)}\right|_{\mathcal{A}} + \left|g^{(N+1)}...h^{(M)}\right|_{\mathcal{A}} = \left|g'g^{(-1)}\right|_{\mathcal{A}} + \left|g''g^{(-1)}\right|_{\mathcal{A}} \geqslant 2R.$$

Therefore $d_{\mathcal{A}}(S_1, S_2) \geqslant R$, which ends the proof. □

5 Some preparatory technical lemmas

Lemma 5.1
Let (Y, d_Y) be a metric space and (X, d_X) be a proper CAT(0) space such that there exists an (L, A)-quasi-isometry $f : Y \to X$. Moreover, let y_n be such sequence of points in Y that there exist a radius $R > 0$ and a positive constant δ such that for every $\varepsilon > 0$ there

exist a point $\widetilde{y} \in Y$ and natural number N such that the ball $B(\widetilde{y}, R)$ $(3LA+\delta)$-separates y_0 from all points y_n for $n > N$ and

$$\frac{1}{d_Y(y_0, \widetilde{y})} < \varepsilon.$$

Then there exists a geodesic ray γ such that $\lim_{n\to\infty} \gamma_{[f(y_0), f(y_n)]} = \gamma$.

Proof. Let us denote $\gamma_n = \gamma_{[f(y_0), f(y_n)]}$. First we will show that

$$\lim_{n\to\infty} d_X(f(y_0), f(y_n)) = \infty.$$

Let $D > 0$. We will show that there is a natural number N such that for every $n > N$ we have $d_X(f(y_0), f(y_n)) > D$. Let $\varepsilon = \frac{1}{L(D+2A)+R}$, let $\widetilde{y} \in Y$ and $N \in \mathbb{N}$ be such that $I = B(\widetilde{y}, R)$ $(3LA + \delta)$-separates y_0 from all points y_n for $n > N$ and $1/d_Y(y_0, \widetilde{y}) < \varepsilon$. Then according to the R-separation lemma the set $K = N_A(f(I))$ separates $f(y_0)$ from $f(y_n)$. Thus the geodesic path between $f(y_0)$ and $f(y_n)$ has to pass through K. Therefore we have the following estimations:

$$d_X(f(y_0), f(y_n)) \geqslant d_X(f(y_0), K) \geqslant \frac{1}{L} d_Y(y_0, I) - 2A$$

$$\geqslant \frac{1}{L}(d(y_0, \widetilde{y}) - R) - 2A > \frac{1}{L\varepsilon} - \frac{R}{L} - 2A = D,$$

and thus $\lim_{n\to\infty} d_X(f(y_0), f(y_n)) = \infty$.

From Lemma 3.2 we conclude that there is a subsequence γ_{n_k} of γ_n such that $\lim_{k\to\infty} \gamma_{n_k} = \gamma$ for a geodesic ray γ. Now we will show that $\lim_{n\to\infty} \gamma_n = \gamma$. Let $t \geqslant 0$ and $\epsilon > 0$ be any constants. We will show that there exists $\widetilde{N}$ such that for any $n > \widetilde{N}$ we have $d_X(\gamma_n(t), \gamma(t)) < \epsilon$. Let $\widetilde{\varepsilon} > 0$ be such that

$$\widetilde{\varepsilon} < \min\left(\frac{\epsilon}{2Lt(2LR + 3A) + \epsilon R + 2LA\epsilon}, \frac{1}{R + 2LA + Lt}\right).$$

Let $\widetilde{N} \in \mathbb{N}$ and $\hat{y} \in Y$ be such that for every $n > \widetilde{N}$ the ball $\widetilde{I} = B(\hat{y}, R)$ $(3LA + \delta)$-separates y_0 from all points y_n, and $\frac{1}{d_Y(y_0, \hat{y})} < \widetilde{\varepsilon}$. Furthermore let k be such that we have $n_k > N$ and $d_X(\gamma_{n_k}(t), \gamma(t)) < \frac{\epsilon}{2}$. We know from the R-separation lemma that for all $n > \widetilde{N}$ the set $\widetilde{K} = N_A(f(\widetilde{I}))$ separates $f(y_0)$ apart from $f(y_n)$, and thus the geodesic paths γ_n and γ_{n_k} have to pass through $\widetilde{K}$. Let s, s' be such that $\gamma_n(s) \in \widetilde{K}$ and $\gamma_{n_k}(s') \in \widetilde{K}$. Then we can estimate:

$$\min(s, s') \geqslant d_X(f(y_0), K) \geqslant d_X(f(y_0), f(\widetilde{I})) - A \geqslant \frac{1}{L} d_Y(y_0, \widetilde{I}) - 2A$$

$$\geqslant \frac{1}{L} d_Y(y_0, \hat{y}) - \frac{R}{L} - 2A > \frac{1}{L\widetilde{\varepsilon}} - \frac{R}{L} - 2A.$$

Since $\widetilde{\varepsilon} < \frac{1}{R+2LA+Lt}$ then we know that $\frac{1}{L\widetilde{\varepsilon}} - \frac{R}{L} - 2A > t$. We can also estimate

$$d_X(\gamma_n(s), \gamma_{n_k}(s')) \leqslant \operatorname{diam}(K) \leqslant \operatorname{diam}(f(I)) + 2A \leqslant L \operatorname{diam}(I) + 3A \leqslant 2LR + 3A.$$

Therefore from convexity of the distance function and triangle cutting lemma we have

$$d_X(\gamma_n(t), \gamma_{n_k}(t)) \leqslant \frac{t}{\min(s, s')} d_X(\gamma_n(\min(s, s')), \gamma_{n_k}(\min(s, s')))$$

$$\leqslant \frac{t}{\min(s,s')} d_X(\gamma_n(s), \gamma_{n_k}(s')) < \frac{t(2LR+3A)}{1/L\widetilde{\varepsilon} - R/L - 2A}.$$

Since $\widetilde{\varepsilon} < \frac{\epsilon}{2Lt(2LR+3A)+\epsilon R + 2LA\epsilon}$ then

$$\frac{t(2LR+3A)}{1/L\widetilde{\varepsilon} - R/L - 2A} < \frac{\epsilon}{2}.$$

Now we have

$$d_X(\gamma_n(t), \gamma(t)) \leqslant d_X(\gamma_n(t), \gamma_{n_k}(t)) + d_X(\gamma_{n_k}(t), \gamma(t)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which ends the proof. □

Lemma 5.2
Let (Y, d_Y) be a metric space and (X, d_X) be a proper CAT(0) space such that there exists an (L, A)-quasi-isometry $f : Y \to X$. Moreover let y_n be such a sequence of points in Y that there exists a geodesic ray γ such that $\gamma = \lim_{n\to\infty} \gamma_{[f(y_0), f(y_n)]}$. Furthermore, let the positive number δ, natural number N and bounded set $I \subseteq Y$ be such that for every $n > N$, I $(3LA + \delta)$-separates y_0 from y_n. Then there exist an $s \in [0, \infty)$ such that $\gamma(s) \in N_A(f(I))$.

Proof. Let $t \geqslant \sup_{y\in I} Ld_Y(y_0, y) + 2A + 1$ be a sufficiently large constant. Moreover we take $n_0 > N$ such that $d_X(\gamma_{[f(y_0), f(y_{n_0})]}(t), \gamma(t)) < 1$. Since

$$\sup_{x\in N_A(f(I))} d_X(f(y_0), x) \leqslant \sup_{y\in I} d_X(f(y_0), f(y)) + A \leqslant L \sup_{y\in I} d_Y(y_0, y) + 2A$$

the entire geodesic segments between $\gamma_{[f(y_0), f(y_{n_0})]}(t)$, $\gamma(t)$ and between $\gamma_{[f(y_0), f(y_{n_0})]}(t)$, $f(y_{n_0})$ do not intersect the set $N_A(f(I))$. Therefore, since the geodesic segments between $\gamma_{[f(y_0), f(y_{n_0})]}(t)$, $\gamma(t)$ and between $\gamma_{[f(y_0), f(y_{n_0})]}(t)$, $f(y_{n_0})$ are outside of $N_A(f(I))$ then $\gamma(t)$ and $f(y_{n_0})$ are in the same connected component of $X \setminus N_A(f(I))$. But from the R-separation lemma we know that the set $N_A(f(I))$ separates $f(y_0)$ from $f(y_{n_0})$, and thus it also separates (y_0) from $\gamma(t)$, therefore there exist an $s \in [0, t]$ such that we have $\gamma(s) \in N_A(f(I))$. □

Lemma 5.3
Let (Y, d_Y) be a metric space and (X, d_X) be a proper CAT(0) space such that there exists an (L, A)-quasi-isometry $f : Y \to X$. Moreover, let y_n and $\widetilde{y}_n$ be such sequences of points in Y that $y_0 = \widetilde{y}_0$ and there exist geodesic rays $\gamma, \widetilde{\gamma}$ such that $\lim_{n\to\infty} \gamma_{[f(y_0), f(y_n)]} = \gamma$ and $\lim_{n\to\infty} \gamma_{[f(\widetilde{y}_0), f(\widetilde{y}_n)]} = \widetilde{\gamma}$. Furthermore let $\delta > 0$, $R > 0$, $N \in \mathbb{N}$ and $\hat{y} \in Y$ be such that for every natural number $n > N$ the ball $I = B(\hat{y}, R)$ $(3LA + \delta)$-separates y_0 from y_n and $\widetilde{y}_0$ from $\widetilde{y}_n$, where $d_Y(y_0, \hat{y}) > R + 2AL$. Then we have the following estimate in Osajda's metric:

$$d_{3A+2RL}(\gamma, \widetilde{\gamma}) \leqslant \frac{L}{d_Y(y_0, \hat{y}) - R - 2AL}$$

Proof. By Lemma 5.2 we know that there exist $s, \widetilde{s} \in [0, \infty)$ such that $\gamma(s), \widetilde{\gamma}(\widetilde{s}) \in K$ where $K = N_A(f(I))$. Therefore by the triangle cutting lemma we know that

$$d_X(\gamma(\min\{s, \widetilde{s}\}), \widetilde{\gamma}(\min\{s, \widetilde{s}\})) \leqslant \operatorname{diam}(K).$$

We now can write the following estimates:

$$\operatorname{diam}(K) \leqslant 2A + \operatorname{diam}(f(I)) \leqslant 3A + 2RL$$

and

$$\min\{s, \widetilde{s}\} \geqslant d_X(f(y_0), K) \geqslant d_X(f(y_0), f(I)) - A \geqslant \frac{1}{L} d_Y(y_0, I) - 2A \geqslant \frac{d_Y(y_0, \hat{y}) - R - 2AL}{L}.$$

Thus from convexity of the distance function we know that

$$\inf\{t \in [0, \infty) : d_X(\gamma(t), \widetilde{\gamma}(t)) \geqslant 3A + 2LR\}) \geqslant \min\{s, \widetilde{s}\} \geqslant \frac{d_Y(y_0, \hat{y}) - R - 2AL}{L}$$

and therefore

$$d_{3A+2RL}(\gamma, \widetilde{\gamma}) \leqslant \frac{L}{d_Y(y_0, \hat{y}) - R - 2AL}.$$

□

Lemma 5.4
Let (Y, d_Y) be a metric space and let (X, d_X) be a proper CAT(0) space such that there exists an (L, A)-quasi-isometry $f : Y \to X$. Moreover, let y_n and $\widetilde{y}_n$ be such sequences of points in Y that $y_0 = \widetilde{y}_0$ and there exist geodesic rays γ and $\widetilde{\gamma}$ such that $\lim_{n\to\infty} \gamma_{[f(y_0), f(y_n)]} = \gamma$ and $\lim_{n\to\infty} \gamma_{[f(\widetilde{y}_0), f(\widetilde{y}_n)]} = \widetilde{\gamma}$. Suppose there exist a ball $I = B(\hat{y}, R)$, a natural number N and a positive constant δ such that for every $n, m > N$ the set I $(3LA + \delta)$-separates y_n from $\widetilde{y}_n$. Then we have $\gamma \neq \widetilde{\gamma}$, and in Osajda's metric d_2 on $\partial_{x_0} X$ we have the following inequality:

$$d_2(\gamma, \widetilde{\gamma}) \geqslant \frac{1}{L d_Y(y_0, \hat{y}) + LR + 2A + 1}.$$

Proof. Let $\hat{y} \in Y$, $R > 0, \delta > 0$ and $N \in \mathbb{N}$ be such that for all $n_1, n_2 \geqslant N$ the set $I = B(\hat{y}, R)$ $(3LA + \delta)$-separates y_{n_1} from y_{n_2}. Then we can estimate:

$$\sup\{d_X(f(y_0), x) : x \in N_A(f(I))\} \leqslant \sup\{d_X(f(y_0), f(y)) : y \in I\} + A$$

$$\leqslant \sup\{L d_Y(y_0, y) + A : y \in I\} + A \leqslant L d_Y(y_0, \hat{y}) + LR + 2A.$$

Let now $t = L d_Y(y_0, \hat{y}) + LR + 2A + 1$ and $\hat{N} \in \mathbb{N}$ will be such that for all $n \geqslant \hat{N}$ we have the following estimates:

$$d_X(\gamma(t), \gamma_{[f(y_0), f(y_n)])}(t)) < 1,$$

$$d_X(\widetilde{\gamma}(t), \gamma_{[f(\widetilde{y}_0), f(\widetilde{y}_n)])}(t)) < 1.$$

From the triangle inequality we can write:

$$d_X(\gamma(t), N_A(f(I))) \geqslant d_X(f(y_0), \gamma(t)) - \sup\{d_X(f(y_0), x) : x \in N_A(f(I))\}$$

$$\geqslant t - L d_Y(y_0, \hat{y}) + LR + 2A = 1,$$

$$d_X(\widetilde{\gamma}(t), N_A(f(I))) \geqslant d_X(f(\widetilde{y}_0), \widetilde{\gamma}(t)) - \sup\{d_X(f(\widetilde{y}_0), x) : x \in N_A(f(I))\}$$

$$\geqslant t - L d_Y(y_0, \hat{y}) + LR + 2A = 1.$$

Therefore geodesic paths between $\gamma_{[f(y_0),f(y_n)]}(t)$ and $f(y_n)$, $\gamma_{[f(\widetilde{y}_0),f(\widetilde{y}_n)]}(t)$ and $f(\widetilde{y}_n)$, $\gamma_{[f(y_0),f(y_n)]}(t)$ and $\gamma(t)$, $\gamma_{[f(\widetilde{y}_0),f(\widetilde{y}_n)]}(t)$ and $\gamma(t)$ are not going through the set $N_A(f(I))$. Therefore $\gamma(t)$ and $f(y_n)$ are in the same pathwise connected component of $X \setminus N_A(f(I))$, and $\widetilde{\gamma}(t)$ and $f(\widetilde{y}_n)$ are in the same pathwise connected component of $X \setminus N_A(f(I))$. Since by the R-separation lemma the set $N_A(f(I))$ separates y_n apart from $\widetilde{y}_n$ in X, then $N_A(f(I))$ is also separates $\gamma(t)$ apart from $\widetilde{\gamma}(t)$ in X. Therefore $\gamma(t) \neq \widetilde{\gamma}(t)$, which implies $\gamma \neq \widetilde{\gamma}$. Moreover

$$d_X(\gamma(t), \widetilde{\gamma}(t)) \geqslant d_X(\gamma(t), N_A(f(I))) + d_X(\widetilde{\gamma}(t), N_A(f(I))) \geqslant 2$$

and thus in Osajda's metric d_2 we have

$$d_2(\gamma, \widetilde{\gamma}) \geqslant \frac{1}{t} = \frac{1}{Ld_Y(y_0, \hat{y}) + LR + 2A + 1}.$$

□

Lemma 5.5
Let (X, d) be a proper CAT(0) space, let $\gamma \in \partial_{x_0} X$ be a geodesic ray, let $x_n \in X$ be a sequence such that $\lim_{n\to\infty} d(\gamma(0), x_n) = \infty$. Suppose that there exists a constant $A \geqslant 0$ such that for any n there exists $t \geqslant 0$ such that $d(\gamma(t), x_n) \leqslant A$. Then $\lim_{n\to\infty} \gamma_{[\gamma(0),x_n]} = \gamma$.

Proof. Let $A > \varepsilon > 0, s \geqslant 0$ be any constants. We will show that for sufficiently large n we have $d(\gamma[\gamma(0), x_n](s), \gamma(s)) < \varepsilon$. Let $N \in \mathbb{N}$ be such that for any $n > N$ we have $d(\gamma(0), x_n) > \frac{sA}{\varepsilon} + A$. We know that there exists a $t \geqslant 0$ such that $d(\gamma(t), x_n) \leqslant A$, thus from triangle inequality we have $t \geqslant d(\gamma(0), x_n) - d(\gamma(t), x_n) > \frac{sA}{\varepsilon}$. For simplicity of notation let $t' = \min\{t, d(\gamma(0), x_n)\}$. Then from convexity and triangle cutting lemma we have

$$d(\gamma(s)\gamma_{[\gamma(0),x_n]}(s)) \leqslant \frac{s}{t'} d(\gamma(t'), \gamma_{[\gamma(0),x_n]}(t'))) < \frac{s}{sA\varepsilon} d(\gamma(t), x_n) \leqslant \varepsilon$$

which ends the proof. □

Lemma 5.6
Let (Y, d_Y) be a metric space and let (X, d_X) be a proper CAT(0) space such that there exists an (L, A)-quasi-isometry $f : Y \to X$. Moreover, let $\gamma \in \partial_{x_0} X$ be a geodesic ray and let $y_n \in Y$ be a sequence such that there exists a geodesic ray $\gamma' \in \partial_{x_0} X$ such that $\gamma' = \lim_{n\to\infty} \gamma_{[f(y_0),f(y_n)]}$. Suppose that there exist a point $\hat{y} \in Y$, radius $R > 0$ and natural number N such that the ball $I = B(\hat{y}, R)$ $(3LA + \delta)$-separates y_0 from y_n for $n > N$. If there exists a number $t \geqslant 6L^2A + 2L\delta + 4A$ such that $d(f(\hat{y}), \gamma(t)) \leqslant A$ then we have the following inequality

$$d_{2RL+4A}(\gamma, \gamma') \leqslant \frac{1}{t - 6L^2A - 2L\delta - 4A}$$

where d_{2RL+4A} denotes the Osajda's metric on $\partial_{x_0} X$.

Proof. From Lemma 5.2 we conclude that there exist $s \in [0, \infty)$ such that we have $\gamma'(s) \in N_A(f(I))$. We can now estimate that:

$$d_X(\gamma(t), \gamma'(s)) \leqslant d_X(\gamma(t), f(\hat{y})) + d_X(f(\hat{y}), \gamma'(s)) \leqslant A + \operatorname{diam}(N_A(f(I)))$$

$$\leqslant 3A + \mathrm{diam}(f(I)) \leqslant 2RL + 4A.$$

Therefore we have $\min\{s,t\} \geqslant t - d_X(\gamma(t), \gamma'(s)) \geqslant t - 2RL - 4A$. From convexity of the distance function and triangle cutting lemma we obtain that

$$d_X(\gamma(t-2RL-4A), \gamma'(t-2RL-4A) \leqslant d_X(\gamma(\min\{s,t\}, \gamma'(\min\{s,t\}))$$
$$\leqslant d_X(\gamma(t), \gamma'(s)) \leqslant 2RL + 4A$$

and thus

$$d_{2RL+4A}(\gamma, \gamma') \leqslant \frac{1}{t - 2RL - 4A}.$$

□

6 Categorisation of geodesic rays

Definition 6.1 (generating sequence)
Let $\Gamma = G * H$ be a free product of groups. We will say that a sequence $k_n \in \Gamma$ is *generating* when it is of one of three types:

(i) there are elements $g_1 \in G$, $g_2, g_3, ... \in G \setminus \{1\}$ and $h_1, h_2, h_3, ... \in H \setminus \{1\}$ such that $k_n = g_1 h_1 g_2 h_2 ... g_n h_n$ for all $n \in \mathbb{N}$;

(ii) there are elements $g_1 \in G$, $g_2, g_3, ..., g_m \in G \setminus \{1\}$, $h_1, h_2, h_3, ..., h_m \in H \setminus \{1\}$ and $\hat{g}_1, \hat{g}_2, \hat{g}_3, ... \in G$ such that $k_n = g_1 h_1 g_2 h_2 ... g_m h_m \hat{g}_n$ for all $n \in \mathbb{N}$;

(iii) there are elements $h_1 \in H$, $h_2, h_3, ..., h_m \in H \setminus \{1\}$, $g_1, g_2, g_3, ..., g_m \in G \setminus \{1\}$ and $\hat{h}_1, \hat{h}_2, \hat{h}_3, ... \in G$ such that $k_n = h_1 g_1 h_2 g_2 ... h_m g_m \hat{h}_n$ for all $n \in \mathbb{N}$.

Definition 6.2 (separated sequences)
Let $\Gamma = G * H$ be a free product of groups. We will say that generating sequences k_n, k'_n are *separated* if one of the following conditions is satisfied:

- k_n, k'_n are both of type (i) and $k_n \neq k'_n$ for some $n \in \mathbb{N}$;
- k_n, k'_n are both of type (ii), $k_n = g_1 h_1 ... g_m h_m \hat{g}_n$ and $k'_n = g'_1 h'_1 ... g'_{m'} h'_{m'} \widetilde{g}_n$ for all $n \in \mathbb{N}$ and either $m \neq m'$ or $g_i h_i \neq g'_i h'_i$ for some $i \in \{1, 2, ..., m\}$;
- k_n, k'_n are both of type (iii), $k_n = h_1 g_1 ... h_m g_m \hat{h}_n$ and $k'_n = h'_1 g'_1 ... h'_{m'} g'_{m'} \widetilde{h}_n$ for all $n \in \mathbb{N}$ and either $m \neq m'$ or $h_i g_i \neq h'_i g'_i$ for some $i \in \{1, 2, ..., m\}$;
- k_n, k'_n are of different types.

Definition 6.3
Let $\Gamma = G * H$ be a free product of groups and let k_n, k'_n be generating sequences. An element $g_1 h_1 ... g_m h_m \in \Gamma$, where $g_1 \in G, h_1, h_2, ..., h_{m-1} \in H \setminus \{1\}, g_2, g_3, ..., g_m \in G \setminus \{1\}, h_m \in H$ is called a *separator* between k_n and k'_n if there exist an $N \in \mathbb{N}$ such that for any natural number $n > N$ one of the sequences k_n, k'_n consists only of prolongations of $g_1 h_1 ... g_m h_m$ and the other consists only of elements that are not prolongations of $g_1 h_1 ... g_m h_m$. Element $g_1 h_1 ... g_m h_m \in \Gamma$ is called the *minimal separator* if it is a separator between k_n and k'_n if it is a separator of minimal "length" m. The element $g_1 h_1 ... g_m h_m \in \Gamma$ is called a *common prefix* of generating sequences k_n, k'_n if there exists an $N \in \mathbb{N}$ such that for every natural $n > N$ both k_n and k'_n are prolongations of $g_1 h_1 ... g_m h_m$.

Lemma 6.4
Let k_n, k'_n be separated generating sequences. Then there exists a unique minimal separator $g_1h_1...g_mh_m$ between k_n and k'_n, and it additionally satisfies the following property: there exists an $N \in \mathbb{N}$ such that for every natural $n > N$ both k_n and k'_n are prolongations of

- *$g_1h_1...h_{m-1}g_m$ if $h_m \neq 1$;*
- *$g_1h_1...g_{m-1}h_{m-1}$ if $h_m = 1$.*

Proof. The proof consist of analyzing many analogical cases. In each of the cases the fact that the additional property from the statement holds is obvious and we omit its justification.

- Both k_n and k'_n are of type (i). Let $k_n = g_1h_1...g_nh_n$, $k'_n = g'_1h'_1...g'_nh'_n$ for all $n \in \mathbb{N}$ and let m be the smallest number such that $k_m \neq k'_m$. If $g_m \neq g'_m$ then $g_1h_1...h_{m-1}g_m$ is the minimal separator. If $g_m = g'_m$ then $g_1h_1...g_mh_m$ is the minimal separator.
- Both k_n and k'_n are of type (ii), $k_n = g_1h_1...g_mh_m\hat{g}_n$ and $k'_n = g'_1h'_1...g'_{m'}h'_{m'}\widetilde{g}_n$. Without loss of generality assume that $m \geqslant m'$. If for all $i \in \{1, 2, ..., m'\}$ we have $g_ih_i = g'_ih'_i$, then $g_1h_1...g_{m'}h_{m'}g_{m'+1}$ is the minimal separator. If there exists an $i \in \{1, ..., m'-1\}$ such that for all $j \in \{1, ..., i\}$ we have $g_jh_j = g'_jh'_j$ and $g_i \neq g'_i$, then $g_1h_1...h_{i-1}g_i$ is the minimal separator. If there exists an $i \in \{1, ..., m'-1\}$ such that for all $j \in \{1, ..., i\}$ we have $g_jh_j = g'_jh'_j$ and $g_i = g'_i$, then $g_1h_1...g_ih_i$ is the minimal separator.
- Both k_n and k'_n are of type (iii), $k_n = h_1g_1...h_mg_m\hat{h}_n$ and $k'_n = h'_1g'_1...h'_{m'}g'_{m'}\widetilde{h}_n$. Without loss of generality assume that $m \geqslant m'$. If for all $i \in \{1, 2, ..., m'\}$ we have $h_ig_i = h'_ig'_i$, then $h_1g_1...h_{m'}g_{m'}h_{m'+1}$ is the minimal separator. If there exists an $i \in \{1, ..., m'-1\}$ such that for all $j \in \{1, ..., i\}$ we have $h_jg_j = h'_jg'_j$ and $h_i \neq h'_i$ then $h_1g_1...g_{i-1}h_i$ is the minimal separator. If there exists an $i \in \{1, ..., m'-1\}$ such that for all $j \in \{1, ..., i\}$ we have $h_jg_j = h'_jg'_j$ and $h_i = h'_i$ then $h_1g_1...h_ig_i$ is the minimal separator.
- One of the k_n, k'_n is of type (i) and the other is of type (ii). Without loss of generality we assume that k_n is of type (i). Let $k_n = g_1h_1...g_nh_n$ for all $n \in \mathbb{N}$ and let $k'_n = g'_1h'_1...g'_mh'_m\hat{g}_n$. If for all $i \in \{1, 2, ..., m'\}$ we have $g_ih_i = g'_ih'_i$, then $g_1h_1...g_{m'}h_{m'}g_{m'+1}$ is the minimal separator. If there exists an $i \in \{1, ..., m'-1\}$ such that for all $j \in \{1, ..., i\}$ we have $g_jh_j = g'_jh'_j$ and $g_i \neq g'_i$ then $g_1h_1...h_{i-1}g_i$ is the minimal separator. If there exists an $i \in \{1, ..., m'-1\}$ such that for all $j \in \{1, ..., i\}$ we have $g_jh_j = g'_jh'_j$ and $g_i = g'_i$ then $g_1h_1...g_ih_i$ is the minimal separator.
- One of the k_n, k'_n is of type (i) and the other is of type (iii). Without loss of generality assume that k_n is of type (i). Let $k_n = g_1h_1...g_nh_n$ for all $n \in \mathbb{N}$ and let $k'_n = h'_1g'_1...h'_{m'}g'_{m'}\hat{h}_n$. If $g_1 \neq 1$ and $h'_1 \neq 1$, then g_1 is the minimal separator. If $g_1 = 1$ and $h'_1 = 1$, then g_1h_1 is the minimal separator. If $g_1 = 1$ but $h'_1 \neq 1$ and for all $i \in \{1, ..., m'\}$ we have $h_ig_{i+1} = h'_ig'_i$, then $g_1h_1...h_{m'}g_{m'+1}$ is the minimal separator. If $g_1 = 1$ but $h'_1 \neq 1$ and there exists an $i \in \{1, ..., m'-1\}$ such that for all $j \in \{1, ..., i\}$ we have $h_jg_{j+1} = h'_jg'_j$ and $h_i \neq h'_i$, then $g_1h_1...g_ih_i$ is the minimal separator. If $g_1 = 1$ but $h'_1 \neq 1$ and there exists an $i \in \{1, ..., m'-1\}$ such that for all $j \in \{1, ..., i\}$ we have $h_jg_{j+1} = h'_jg'_j$ and $h_i = h'_i$, then $g_1h_1...h_ig_{i+1}$ is the minimal separator. If $g_1 \neq 1$ but $h'_1 = 1$ and for all $i \in \{1, ..., m'-1\}$ we have

$g_ih_i = g'_ih'_{i+1}$ and $g_{m'} = g'_{m'}$, then $g_1h_1...g_{m'}h_{m'}$ is the minimal separator. If $g_1 \neq 1$ but $h'_1 = 1$ and for all $i \in \{1, ..., m'-1\}$ we have $g_ih_i = g'_ih'_{i+1}$ and $g_{m'} \neq g'_{m'}$, then $g_1h_1...h_{m'-1}g_{m'}$ is the minimal separator. If $g_1 \neq 1$ but $h'_1 = 1$ and there exist $i \in \{1, ..., m'-2\}$ such that for all $j \in \{1, ..., i\}$ we have $g_jh_j = g'_jh'_{j+1}$ and $g_{i+1} \neq g'_{i+1}$, then $g_1h_1...h_ig_{i+1}$ is the minimal separator. If $g_1 \neq 1$ but $h'_1 = 1$ and there exist $i \in \{1, ..., m'-2\}$ such that for all $j \in \{1, ..., i\}$ we have $g_jh_j = g'_jh'_{j+1}$ and $g_{i+1} = g'_{i+1}$, then $g_1h_1...g_{i+1}h_{i+1}$ is the minimal separator.

- One of the k_n, k'_n is of type (ii) and the other is of type (iii). Without loss of generality assume that k_n is of type (ii). Let $k_n = g_1h_1...g_mh_m\hat{g}_n$ and $h'_1g'_1...h'_{m'}g'_{m'}\widetilde{h}_n$ for all $n \in \mathbb{N}$. Without loss of generality assume that $m \geqslant m'$. If $g_1 \neq 1$ and $h'_1 \neq 1$, then g_1 is the minimal separator. If $g_1 = 1$ and $h'_1 = 1$, then g_1h_1 is the minimal separator. If $g_1 = 1$ but $h'_1 \neq 1$ and for all $i \in \{1, ..., m'\}$ we have $h_ig_{i+1} = h'_ig'_i$, then $g_1h_1...h_{m'}g_{m'+1}$ is the minimal separator. If $g_1 = 1$ but $h'_1 \neq 1$ and there exists an $i \in \{1, ..., m'-1\}$ such that for all $j \in \{1, ..., i\}$ we have $h_jg_{j+1} = h'_jg'_j$ and $h_i \neq h'_i$, then $g_1h_1...g_ih_i$ is the minimal separator. If $g_1 = 1$ but $h'_1 \neq 1$ and there exists an $i \in \{1, ..., m'-1\}$ such that for all $j \in \{1, ..., i\}$ we have $h_jg_{j+1} = h'_jg'_j$ and $h_i = h'_i$, then $g_1h_1...h_ig_{i+1}$ is the minimal separator. If $g_1 \neq 1$ but $h'_1 = 1$ and for all $i \in \{1, ..., m'-1\}$ we have $g_ih_i = g'_ih'_{i+1}$ and $g_{m'} = g'_{m'}$, then $g_1h_1...g_{m'}h_{m'}$ is the minimal separator. If $g_1 \neq 1$ but $h'_1 = 1$ and for all $i \in \{1, ..., m'-1\}$ we have $g_ih_i = g'_ih'_{i+1}$ and $g_{m'} \neq g'_{m'}$, then $g_1h_1...h_{m'-1}g_{m'}$ is the minimal separator. If $g_1 \neq 1$ but $h'_1 = 1$ and there exists an $i \in \{1, ..., m'-2\}$ such that for all $j \in \{1, ..., i\}$ we have $g_jh_j = g'_jh'_{j+1}$ and $g_{i+1} \neq g'_{i+1}$, then $g_1h_1...h_ig_{i+1}$ is the minimal separator. If $g_1 \neq 1$ but $h'_1 = 1$ and there exists an $i \in \{1, ..., m'-2\}$ such that for all $j \in \{1, ..., i\}$ we have $g_jh_j = g'_jh'_{j+1}$ and $g_{i+1} = g'_{i+1}$, then $g_1h_1...g_{i+1}h_{i+1}$ is the minimal separator.

□

Lemma 6.5
Let $\Gamma = G * H$ be a free product of non-trivial groups and let $\Gamma \curvearrowright X$ for a CAT(0) space X. Moreover let $g_1 \in G$, $g_2, g_3, ... \in G \setminus \{1\}$ and $h_1, h_2, h_3, ... \in H \setminus \{1\}$. Then there exists a geodesic ray $\gamma \in \partial_{x_0}X$ such that $\gamma = \lim_{n\to\infty} \gamma_{[x_0, g_1h_1g_2h_2...g_nh_n \cdot x_0]}$ does exist.

Proof. Let $\mathcal{A}_G$ and $\mathcal{A}_H$ be finite sets of generators of G and H respectively, and let $\mathcal{A} = \mathcal{A}_G \sqcup \mathcal{A}_H$. From Švarc-Milnor lemma we know that there exists an (L, A)-quasi-isometry $f : \Gamma \to X$ between $(\Gamma, d_{\mathcal{A}})$ and X. Let ε, δ be any positive constants and let $M > \max\left\{\frac{1}{2\varepsilon}, 3LA + \delta\right\}$. From fact 2.30 we know that

$$d_{\mathcal{A}}(1, g_1h_1g_2h_2...g_Mh_M) \geqslant 2M > 6LA + \delta \quad \text{and} \quad 1/d_{\mathcal{A}}(1, g_1h_1g_2h_2...g_Mh_M) < \frac{1}{2M} < \varepsilon.$$

Moreover for all $n \geqslant 2M$ we have

$$d_{\mathcal{A}}(g_1h_1...g_Mh_M, g_1h_1...g_nh_n) = d_{\mathcal{A}}(1, g_{M+1}h_{M+1}...g_nh_n) \geqslant 2n - 2M \geqslant 2M > 6LA + 2\delta.$$

Therefore from Lemma 5.1 the limit $\lim_{n\to\infty} \gamma_{[x_0, g_1h_1g_2h_2...g_nh_n \cdot x_0]}$ does exist. □

Lemma 6.6
Let $\Gamma = G * H$ be a free product of groups and let $\Gamma \curvearrowright X$ for a proper CAT(0) space (X, d). Moreover let $x_0 \in X$ be any point and let k_n, k'_n be separated generating sequences such

that the limits $\lim_{n\to\infty} \gamma_{[x_0,k_n\cdot x_0]} = \gamma$, $\lim_{n\to\infty} \gamma_{[x_0,k'_n\cdot x_0]} = \gamma'$ *for geodesic rays* $\gamma, \gamma' \in \partial_{x_0} X$ *respectively. Then we have* $\gamma \neq \gamma'$*.*

Proof. From Lemma 6.4 we know that there exists a separator between k_n and k'_n, so let $g \in \Gamma$ be any such separator between k_n and k'_n. Since $\lim_{n\to\infty} \gamma_{[x_0,k_n\cdot x_0]}$ and $\lim_{n\to\infty} \gamma_{[x_0,k'_n\cdot x_0]}$ do exist, $\lim_{n\to\infty} d(x_0, k_n \cdot x_0) = \lim_{n\to\infty} d(x_0, k'_n \cdot x_0) = \infty$. From Švarc-Milnor lemma we know that for a finite set of generators $\mathcal{A}$ of Γ, the spaces (X, d) and $(\Gamma, d_{\mathcal{A}})$ are (L, A)-quasi-isometric, and thus $\lim_{n\to\infty} d_{\mathcal{A}}(1, k_n) = \lim_{n\to\infty} d_{\mathcal{A}}(1, k'_n) = \infty$. Therefore $\lim_{n\to\infty} d_{\mathcal{A}}(g, k_n) = \lim_{n\to\infty} d_{\mathcal{A}}(g, k'_n) = \infty$. Let $R > 3LA$ and let $N \in \mathbb{N}$ be such that for all natural numbers $n > N$ we have $d_{\mathcal{A}}(g, k_n) \geqslant 2R$ and $d_{\mathcal{A}}(g, k'_n) \geqslant 2R$. Therefore from Lemma 4.6 the ball $I = B(g, R)$ R-separaes k_n from $k'_{n'}$ for all $n, n' > N$. From Lemma 5.4 we conclude that $\gamma \neq \gamma'$. □

Lemma 6.7

Let $\Gamma = G * H$ *be a free product of nontrivial groups and let* $\Gamma \curvearrowright X$ *for a proper CAT(0) space* (X, d)*. Moreover let* γ *be a geodesic ray beginning at* x_0*. Then* $\gamma = \lim_{n\to\infty} \gamma_{[x_0,k_n\cdot x_0]}$ *for some generating sequence* k_n*.*

Proof. Let $\gamma \in \partial_{x_0} X$ be any geodesic ray. Moreover let $\mathcal{A}_G$, $\mathcal{A}_H$ be any finite sets generating G, H respectively. From Švarc-Milnor lemma we know that $f : x_0 \mapsto g \cdot x_0$ is an (L, A)-quasi-isometry between $(\Gamma, d_{\mathcal{A}})$ and (X, d) and thus $\Gamma \cdot x_0$ is A-quasi-dense in X. We define the set $C \subseteq \Gamma$ as

$$C = \{g \in \Gamma : \exists t \geqslant 0 \; d(g \cdot x_0, \gamma(t) \leqslant A)\} .$$

Since for every $t \geqslant 0$ there exists a $g \in \Gamma$ such that $d(\gamma(t), g \cdot x_0)$ and for any t, t' such that $t + 2A > t'$ there cannot be any point $x \in X$ such that $d(\gamma(t), x) \leqslant A$ and $d(\gamma(t'), x) \leqslant A$ then set C is infinite.

Let $g \in \Gamma$ be such an element that infinitely many elements in C are prolongations of g. We will show that there exists at most one element g' which is such a prolongation of g that $g^{-1}g' \in G \cup H$ and infinitely many elements in C are prolongations of g'. Assume on the contrary that there are two such elements g', g''. Then g'' is not a prolongation of g' and g' is not a prolongation of g''. We will now define

$$C' = \{g \in C : g \text{ is a prolongation of } g'\}$$

$$C'' = \{g \in C : g \text{ is a prolongation of } g''\} .$$

Since sets C' and C'' are unbounded, we conclude from Lemma 3.2 that there are sequences $c'_n \in C'$ and $c''_n \in C''$ such that $\lim_{n\to\infty} \gamma_{[x_0,c'_n\cdot x_0]} = \gamma'$ and $\lim_{n\to\infty} \gamma_{[x_0,c''_n\cdot x_0]} = \gamma''$ for some geodesic rays γ' and γ''. Moreover for any $\delta > 0$ and sufficiently large n', n'' we have $d_{\mathcal{A}}(g', c'_{n'}) \geqslant 6LA + 2\delta$ and $d_{\mathcal{A}}(g', c''_{n''}) \geqslant 6LA + 2\delta$, thus from Lemma 4.6 the set $I = B(g', 3LA + \delta)$ $(3LA + \delta)$-separates $c'_{n'}$ from $c''_{n''}$. Therefore from Lemma 5.4 $\gamma' \neq \gamma''$. On the other hand however, we know from Lemma 5.5 that $\lim_{n\to\infty} \gamma_{[x_0,c'_n\cdot x_0]} = \lim_{n\to\infty} \gamma_{[x_0,c''_n\cdot x_0]} = \gamma$. The contradiction ends this parts of the proof.

Now we need to consider two cases:

- There are infinite sequences of elements $g_1 \in G$, $h_1, h_2, ... \in H \backslash \{1\}$, $g_2, g_3, ... \in G \setminus \{1\}$ such that for every element $k_n = g_1 h_1 ... g_n h_n$, infinitely many elements from C are prolongations of k_n. Let $\varepsilon > 0$ be any positive number, let $m \geqslant 6LA + 2\varepsilon$ be a natural number and let c_n be any sequence consisting of all points in C. From Lemma 5.5 we know that $\lim_{n\to\infty} \gamma_{x_0, c_n \cdot x_0} = \gamma$. Moreover, for any $m \in \mathbb{N}$ there are $\varepsilon > 0, N \in \mathbb{N}$ such that for any natural number $n > N$ the element c_n is a prolongation of k_m and $d_{\mathcal{A}}(c_n, k_m) \geqslant 6LA + 2\varepsilon$. Therefore from Lemma 4.6 the ball $I_m = B(k_m, 3LA + \varepsilon)$ $(3LA + \varepsilon)$-separates 1 from c_n, and thus from Lemma 5.2 there exists an $s_m \geqslant 0$ such that $\gamma(s_m) \in N_A(f(I_m))$. From the definition of (L, A)-quasi-isometry we get the following estimate

$$d(\gamma(s_m), k_m \cdot x_0) \leqslant \operatorname{diam}(N_A(f(I_m))) \leqslant 2A + \operatorname{diam}(f(I_m)) \leqslant 6L^2A + 2L\varepsilon + 3A.$$

Therefore from Lemma 5.5 we know that

$$\lim_{n\to\infty} \gamma_{[x_0, k_n \cdot x_0]} = \gamma.$$

- There is an element $k_0 \in \Gamma$ such that infinitely many elements of C are prolongations of k_0, but there are no prolongations k_0' of k_0 such that in C there are infinitely many elements that are prolongations of k_0'. Without loss of generality we can assume that k_0 is of the form $g_1h_1...g_mh_m$ for $g_1 \in G, h_1, h_2, ..., h_m \in H \setminus \{1\}$ and $g_2, ..., g_m \in G \setminus \{1\}$. Let c_n be a sequence consisting of all prolongations of k_0 that are in C and let $\hat{g}_n$ be such that c_n is either equal to, or is a prolongation of $k_0\hat{g}_n$. Suppose that there exists an $\varepsilon > 0$ such that there are infinitely many c_n such that $d_{\mathcal{A}}(c_n, k_0\hat{g}_n) \leqslant 6LA + 2\varepsilon$. Let t_n be such that $d(c_n \cdot x_0, \gamma(t_n)) \leqslant A$, then we have

$$d(\gamma(t_n), k_0\hat{g}_n \cdot x_0) \leqslant d(\gamma(t_n), c_n \cdot x_0) + d(c_n \cdot x_0, k_0\hat{g}_n \cdot x_0) \leqslant 6L^2A + 2L\varepsilon + 2A.$$

Since this estimate does not depend on the choice of n then from Lemma 5.5 we have

$$\lim_{n\to\infty} \gamma_{[x_0, k_0\hat{g}_n \cdot x_0]} = \gamma.$$

Now suppose that there are only finitely many c_n such that $d_{\mathcal{A}}(c_n, k_0\hat{g}_n) \leqslant 6LA+2\varepsilon$. From Lemma 5.5 we know that $\lim_{n\to\infty} \gamma_{[x_0, c_n \cdot x_0]} = \gamma$, thus for a given $s > 0$ and $\frac{1}{[6L^2A+2L\varepsilon+3A]L} > \epsilon > 0$ let $N \in \mathbb{N}$ be such that for all natural $n > N$ we have $d(\gamma(s), \gamma_{[x_0, c_n \cdot x_0]}(s)) \leqslant \epsilon$,

$$|k_0\hat{g}_n|_{\mathcal{A}} \geqslant \frac{[6L^2A + 2L\varepsilon + 3A]Ls}{\epsilon} + 4LA + 6L^3A + 2L^2\varepsilon$$

and $d_{\mathcal{A}}(c_n, k_0\hat{g}_n) > 6LA + 2\varepsilon$. Then from Lemma 4.6 the ball $I = B(k_0\hat{g}_n, 3LA + \varepsilon)$ $(3LA + \varepsilon)$-separates 1 from c_n in $(\Gamma, d_{\mathcal{A}})$. From the R-separation lemma we know that the set $K = N_A(f(I))$ separates x_0 from $c_n \cdot x_0$, thus there is a t such that $\gamma_{[x_0, c_n \cdot x_0]}(t) \in K$. Now we have the following estimates:

$$\operatorname{diam}(K) \leqslant 6L^2A + 2L\varepsilon + 3A$$

and

$$\min\{t, d(x_0, k_0\hat{g}_n)\} \geqslant \frac{|k_0\hat{g}_n|_{\mathcal{A}}}{L} - A - \operatorname{diam}(K) \geqslant \frac{|k_0\hat{g}_n|_{\mathcal{A}}}{L} - 4A - 6L^2A - 2L\varepsilon$$

$$\geqslant \frac{[6L^2A + 2L\varepsilon + 3A]s}{\epsilon}.$$

For simplicity we will denote $t' = \min\{t, d(x_0, k_0\hat{g}_n)\}$. From convexity of distance function and triangle cutting lemma we have

$$d(\gamma(s), \gamma_{[x_0,k_0\hat{g}_n\cdot x_0]}(s)) \leqslant d(\gamma(s), \gamma_{[x_0,c_n\cdot x_0]}(s)) + d(\gamma_{[x_0,c_n\cdot x_0]}(s), \gamma_{[x_0,k_0\hat{g}_n\cdot x_0]}(s))$$
$$< \epsilon + \frac{s}{t'} d(\gamma_{[x_0,c_n\cdot x_0]}(t'), \gamma_{[x_0,k_0\hat{g}_n\cdot x_0]}(t')) \leqslant \epsilon + \frac{s}{t'} d(\gamma_{[x_0,c_n\cdot x_0]}(t), k_0\hat{g}_n \cdot x_0)$$
$$\leqslant \varepsilon + \frac{s}{t'} \operatorname{diam}(K) \leqslant 2\varepsilon.$$

Therefore $\lim_{n\to\infty} \gamma_{[x_0,k_0\hat{g}_n\cdot x_0]} = \gamma$ which ends the proof.

□

Lemma 6.8
Let $\Gamma = G*H$ be a free product of groups, let $\Gamma \curvearrowright X$ for a proper CAT(0) space (X, d) and let $x_0 \in X$. Moreover let k_n, k'_n be such generating sequences that $\lim_{n\to\infty} \gamma_{[x_0,k_n\cdot x_0]} = \gamma$ and $\lim_{n\to\infty} \gamma_{[x_0,k'_n\cdot x_0]} = \gamma'$ for geodesic rays γ, γ'. Let k be a common prefix of k_n, k'_n. Furthermore let $\mathcal{A}$ be any finite set of generators of Γ. There exists a constant $R > 0$ (depending only on Γ, X, x_0 and $\mathcal{A}$) such that for every $\varepsilon > 0$ there exists a constant $N \in \mathbb{N}$ such that if $|k|_{\mathcal{A}} \geqslant N$ then $d_R(\gamma, \gamma') < \varepsilon$ for an Osajda's metric d_R on $\partial_{x_0} X$.

Proof. From Švarc-Milnor lemma we know that $f : g \mapsto g \cdot x_0$ is an (L, A)-quasi-isometry between $(\Gamma, d_{\mathcal{A}})$ and (X, d) for some $L \geqslant 1, A \geqslant 0$. Let $\delta > 0$ be any number, let $R = 3A + 2(3AL + \delta)L$ and for given $\varepsilon > 0$ let

$$N > \max\left\{\frac{L}{\varepsilon} + 5LA + \delta, 2(3AL + \delta)\right\}$$

be a natural number. Since $|k|_{\mathcal{A}} \geqslant N > 2(3AL + \delta)$ and $|k_n|_{\mathcal{A}} \to \infty, |k'_n|_{\mathcal{A}} \to \infty$, then there exists an $M \in \mathbb{N}$ such that for any natural number $n > M$ from Lemma 4.6 the set $I = B(k, 3LA + \delta)$ $(3LA + \delta)$-separates 1 form k_n and 1 from k'_n. Therefore from Lemma 5.3 we have the following estimate:

$$d_R(\gamma, \gamma') \leqslant \frac{L}{|k|_{\mathcal{A}} - 3LA - \delta - 2LA} < \varepsilon,$$

which ends the proof. □

Lemma 6.9
Let $\Gamma = G * H$ be a free product of infinite groups, let $\Gamma \curvearrowright X$ for a proper CAT(0) space (X, d) and let $x_0 \in X$. Moreover let $\gamma \in \partial_{x_0} X$ be any geodesic ray and let $\mathcal{A}$ be any finite set of generators of Γ. There exist a constant $R > 0$ such that for every $\varepsilon > 0$ there exist generating sequences $k_n^{(i)}, k_n^{(ii)}, k_n^{(iii)}$ of type (i), (ii), (iii) respectively such that $\lim_{n\to\infty} \gamma_{[x_0,k_n^{(i)}\cdot x_0]} = \gamma^{(i)}$, $\lim_{n\to\infty} \gamma_{[x_0,k_n^{(ii)}\cdot x_0]} = \gamma^{(ii)}$, $\lim_{n\to\infty} \gamma_{[x_0,k_n^{(iii)}\cdot x_0]} = \gamma^{(iii)}$ for geodesic rays $\gamma^{(i)}, \gamma^{(ii)}, \gamma^{(iii)}$ and $d_R(\gamma, \gamma^{(i)}) < \varepsilon, d_R(\gamma, \gamma^{(ii)}) < \varepsilon, d_R(\gamma, \gamma^{(iii)}) < \varepsilon$ for an Osajda's metric d_R on $\partial_{x_0} X$.

Proof. From Švarc-Milnor lemma we know that $f : g \mapsto g \cdot x_0$ is an (L, A)-quasi-isometry between $(\Gamma, d_{\mathcal{A}})$ and (X, d) for some $L \geqslant 1, A \geqslant 0$. Let $\delta > 0$ and let $t > \frac{1}{\varepsilon} + 6L^2A + 2L\delta + 4A$ be a sufficiently large positive constant. Since from the definition of (L, A)-quasi-isometry the set $\Gamma \cdot x_0$ is quasi-dense in X, then let $g \in \Gamma$ be such that $d(k \cdot x_0, \gamma'(t)) \leqslant A$. Let $g \in G \setminus \{1\}$, $h \in H \setminus \{1\}$ be fixed elements and let $\hat{g}_n \in G, \hat{h}_n \in H$ be any two sequences of elements of G, H such that $\hat{g}_n \neq \hat{g}_m$ and $\hat{h}_n \neq \hat{h}_m$ for $n \neq m$. Note that since sequences $\hat{g}_n$, $\hat{h}_n$ consist of pairwise distinct elements then $\lim_{n\to\infty} |\hat{g}_n|_{\mathcal{A}} = \lim_{n\to\infty} \left|\hat{h}_n\right|_{\mathcal{A}} = \infty$. Let us consider two cases:

- k has normal form $g_1 h_1..g_m h_m$, where $h_m \neq 1$. Then we define

$$k_n^{(i)} = \begin{cases} g_1 h_1 ... g_n h_n \text{ for } n \leqslant m \\ g_1 h_1 ... g_m h_m (gh)^{n-m} \text{ for } n > m, \end{cases}$$

  $k_n^{(ii)}$ is such a subsequence of $g_1 h_1 ... g_m h_m \hat{g}_n$ that there exists a geodesic ray $\gamma^{(ii)}$ such that $\gamma^{(ii)} = \lim_{n\to\infty} \gamma_{[x_0, k_n^{(ii)} \cdot x_0]}$, $k_n^{(iii)}$ is such a subsequence of $g_1 h_1 ... g_m h_m g \hat{h}_n$ that there exists a geodesic ray $\gamma^{(iii)}$ such that $\gamma^{(iii)} = \lim_{n\to\infty} \gamma_{[x_0, k_n^{(iii)} \cdot x_0]}$;

- k has normal form $g_1 h_1..g_m h_m$, where $h_m = 1$. Then we define

$$k_n^{(i)} = \begin{cases} g_1 h_1 ... g_n h_n \text{ for } n < m \\ g_1 h_1 ... g_m h \text{ for } n = m \\ g_1 h_1 ... g_m h (gh)^{n-m} \text{ for } n > m, \end{cases}$$

  $k_n^{(ii)}$ is such a subsequence of $g_1 h_1 ... g_m h \hat{g}_n$ that there exists a geodesic ray $\gamma^{(ii)}$ such that $\gamma^{(ii)} = \lim_{n\to\infty} \gamma_{[x_0, k_n^{(ii)} \cdot x_0]}$, $k_n^{(iii)}$ is such a subsequence of $g_1 h_1 ... g_m \hat{h}_n$ that there exists a geodesic ray $\gamma^{(iii)}$ such that $\gamma^{(iii)} = \lim_{n\to\infty} \gamma_{[x_0, k_n^{(iii)} \cdot x_0]}$.

In both cases all of the sequences $k_n^{(i)}, k_n^{(ii)}, k_n^{(iii)}$ consist only of prolongations of k for $n > m$. Therefore from Lemma 4.6 we know that there exist $N \in \mathbb{N}$ such that for natural $n > N$ the ball $I = (k, 3LA + \delta)$ $(3LA + \delta)$-separates 1 form $k_n^{(i)}, k_n^{(ii)}$ and $k_n^{(iii)}$. Therefore from Lemma 5.6 we know that for $R = 6L^2 A + 2L\delta + 4A$ we have the following inequalities in Osajda's metric d_R:

$$d_R(\gamma, \gamma^{(i)}) < \varepsilon, \qquad d_R(\gamma, \gamma^{(ii)}) < \varepsilon, \qquad d_R(\gamma, \gamma^{(iii)}) < \varepsilon.$$

□

Lemma 6.10
Let $\Gamma = G * H$ be a free product of groups, let $\Gamma \curvearrowright X$ for a proper CAT(0) space (X, d) and let $x_0 \in X$. Moreover let $\mathcal{A}$ be any finite set of generators of Γ and let k_n, k_n' be such generating sequences that $\lim_{n\to\infty} \gamma_{[x_0, k_n \cdot x_0]} = \gamma$ and $\lim_{n\to\infty} \gamma_{[x_0, k_n' \cdot x_0]} = \gamma'$ for geodesic rays γ, γ'. Let k be a separator of k_n, k_n'. There exist constants $a \geqslant 1, b \geqslant 1$ depending only on $\Gamma, \mathcal{A}, X$ and x_0, such that $d_2(\gamma, \gamma') > \frac{1}{a|k|_{\mathcal{A}} + b}$, where d_2 is an Osajda's metric on $\partial_{x_0} X$.

Proof. Let $f : g \mapsto g \cdot x_0$ be an (L, A)-quasi-isometry between $(\Gamma, d_{\mathcal{A}})$ and (X, d) and let $a = L$, $b = L(3LA + \delta) + 2A + 1$. From the definition of a separator and Lemma 4.6 we know that there exists a natural number N such that the set $I = B(k, 3LA + \delta)$ $(3LA + \delta)$-separates k_n from $k_{n'}$ for $n, n' > N$. Therefore from Lemma 5.4 we know that $d_2(\gamma, \gamma') \geqslant \frac{1}{a|k|_{\mathcal{A}} + b}$, where d_2 is an Osajda's metric on $\partial_{x_0} X$. □

7 Proof of the main theorem

Theorem 7.1
Let $\Gamma = G * H$ be a free product of infinite groups such that $\Gamma \curvearrowright X$ for a proper CAT(0) space (X, d). Then ∂X can be expressed in terms of ΛG and ΛH in the following way:

$$\partial X \cong \widetilde{\sqcup}(\Lambda G, \Lambda H).$$

Proof. The proof will naturally split into five parts, each corresponding to the respective axiom of the dense amalgam as in definition 2.34. Let $\mathcal{Y}_G = \{\Lambda(kG) : k \in \Gamma\}$, $\mathcal{Y}_H = \{\Lambda(kH) : k \in \Gamma\}$ and let $\mathcal{Y} = \mathcal{Y}_G \sqcup \mathcal{Y}_H$ be distinguished families of subspaces of ∂X. Moreover let $\mathcal{A}_G$, $\mathcal{A}_H$ be finite sets of generators of G, H respectively and let $\mathcal{A} = \mathcal{A}_G \sqcup \mathcal{A}_H$.

(i) From Lemma 3.8 we know that for each $\Lambda(kG), \Lambda(k'G) \in \mathcal{Y}_G$ and $\Lambda\hat{k}H, \Lambda\hat{k}'H \in \mathcal{Y}_H$ we have $\Lambda(kG) \cong \Lambda(k'G)$ and $\Lambda(\hat{k}H) \cong \Lambda(\hat{k}'H)$. Let $Y_1, Y_2 \in \mathcal{Y}$ be two different copies of ΛG or ΛH. Then we need to consider two subcases:

a) One of the copies Y_1, Y_2 is a copy of ΛG and the other is a copy of ΛH. Without loss of generality let $Y_1 = \Lambda kG$ and $Y_2 = \Lambda(k'H)$. Assume on the contrary that there exists a $\xi \in \Lambda(kG)$ such that $\xi \in \Lambda(k'H)$. Then there is $\gamma \in \Lambda_{x_0}(kG \cdot x_0)$ such that $\gamma \in \Lambda_{x_0}(k'H \cdot x_0)$. From the definition of limit set we obtain

$$\gamma = \lim_{n\to\infty} \gamma_{[x_0, kg_n \cdot x_0]} = \lim_{n\to\infty} \gamma_{[x_0, k'h_n \cdot x_0]}$$

but $c_n = kg_n$ and $c'_n = k'h_n$ are separated generating sequences. Therefore from Lemma 6.6 we obtain $\lim_{n\to\infty} \gamma_{[x_0, kg_n \cdot x_0]} \neq \lim_{n\to\infty} \gamma_{[x_0, k'h_n \cdot x_0]}$, a contradiction.

b) Both Y_1 and Y_2 are copies of ΛG or of ΛH. Without loss of generality assume that $Y_1 = \Lambda(kG)$ and $Y_2 = \Lambda(k'G)$ where $k \neq k'g$ for every $g \in G$. Assume on the contrary that there exists a $\xi \in \Lambda(kG)$ such that $\xi \in \Lambda(k'G)$. Then there is a $\gamma \in \Lambda_{x_0}(kG \cdot x_0)$ such that $\gamma \in \Lambda_{x_0}(k'G \cdot x_0)$. From the definition of limit set we obtain $\gamma = \lim_{n\to\infty} \gamma_{[x_0, kg_n \cdot x_0]} = \lim_{n\to\infty} \gamma_{[x_0, k'g'_n \cdot x_0]}$, but $c_n = kg_n$ and $c'_n = k'g_n$ are separated generating sequences. Therefore from Lemma 6.6 we obtain

$$\lim_{n\to\infty} \gamma_{[x_0, kg_n \cdot x_0]} \neq \lim_{n\to\infty} \gamma_{[x_0, k'g_n \cdot x_0]}.$$

(ii) We know from Lemma 2.32 that it is enough to show that the family $\mathcal{Y}$ is null with respect to any preferred metric d compatible with the topology on ∂X. We will show that both families $\mathcal{Y}_G$ and $\mathcal{Y}_H$ are null. Without loss of generality we need only to show that for the former. Let $x_0 \in X$ and let R be a constant from Lemma 6.8 for the boundary $\partial_{x_0} X$. Suppose on the contrary that there exist an infinite sequence of cosets $g_1^{(n)} h_1^{(n)} ... g_{m^{(n)}}^{(n)} h_{m^{(n)}}^{(n)} G$ such that $\mathrm{diam}_R(\Lambda_{x_0}(g_1^{(n)} h_1^{(n)} ... g_{m^{(n)}}^{(n)} h_{m^{(n)}}^{(n)} G \cdot x_0)) > \varepsilon$ where diam_R is a diameter in Osajda's metric d_R. Moreover, for the ε given above, let N be the constant from Lemma 6.8. Since the set of generators $\mathcal{A}$ is finite, there are only finitely many words $g_1^{(n)} h_1^{(n)} ... g_{m^{(n)}}^{(n)} h_{m^{(n)}}^{(n)}$ such that $\left| g_1^{(n)} h_1^{(n)} ... g_{m^{(n)}}^{(n)} h_{m^{(n)}}^{(n)} \right|_{\mathcal{A}} < N$. Let $n_0 \in \mathbb{N}$ be such that $\left| g_1^{(n_0)} h_1^{(n_0)} ... g_{m^{(n_0)}}^{(n_0)} h_{m^{(n_0)}}^{(n_0)} \right|_{\mathcal{A}} \geqslant N$. For the sake of simplifying notation we will denote $k = g_1^{(n_0)} h_1^{(n_0)} ... g_{m^{(n_0)}}^{(n_0)} h_{m^{(n_0)}}^{(n_0)}$. Then let $\gamma, \gamma' \in \Lambda_{x_0}(kG \cdot x_0)$ be any geodesic rays and let $\gamma = \lim_{n\to\infty} \gamma_{[x_0, kg_n \cdot x_0]}$, $\gamma' =$

$\lim_{n\to\infty}\gamma_{[x_0,kg'_n\cdot x_0]}$, where $g_n, g'_n \in G$. Note that kg_n, kg'_n are generating sequences and k is their common prefix. From Lemma 6.8 we get $d_R(\gamma,\gamma') < \varepsilon$, thus

$$\operatorname{diam}(\Lambda_{x_0}(kG\cdot x_0)) \leqslant \varepsilon.$$

The contradiction proves that the family $\mathcal{Y}_G$ is null.

(iii) Let $Y \in \mathcal{Y}$ be any of the embedded copies of ΛG or ΛH, without loss of generality we will assume that $Y = \Lambda(g_1h_1...g_mh_mG)$. Moreover let R be the constant from Lemma 5.9, let $\gamma' \in \Lambda_{x_0}(g_1h_1...g_mh_mG\cdot x_0)$ and let $\varepsilon > 0$ be any positive number. Then from Lemma 6.9 we know that there exists a generating sequence k_n of type (i) such that $\gamma = \lim_{n\to\infty}\gamma_{[x_0,k_n\cdot x_0]}$ and $d_R(\gamma,\gamma') < \varepsilon$ for an Osajda's metric d_R on $\partial_{x_0}X$. The generating sequence k_n is separated from any generating sequence of type (ii), thus it is separated from all generating sequences $k'_n = g_1h_1...g_mh_m\hat{g}_n$, where $\hat{g}_n \in G$. Therefore from Lemma 6.6 $\gamma \neq \gamma''$ for any γ'' given as $\lim_{n\to\infty}\gamma_{[x_0,g_1h_1...g_mh_m\hat{g}_n\cdot x_0]}$, so $\gamma \notin \Lambda_{x_0}(g_1h_1...g_mh_mG\cdot x_0)$. From the arbitrariness of the choice of ε we conclude that Y is a boundary subset.

(iv) Without loss of generality we will show that $\bigcup\mathcal{Y}_G$ is dense. Let $R > 0$ be the constant from Lemma 6.9 and let $\gamma \in \partial_{x_0}X$ be any geodesic ray. Then it follows from Lemma 6.9 that for any $\varepsilon > 0$ there exists a geodesic ray $\gamma^{(ii)}$ such that $d_R(\gamma,\gamma^{(ii)}) < \varepsilon$ in Osajda's metric d_R on $\partial_{x_0}X$ and $\gamma^{(ii)} = \lim_{n\to\infty}\gamma_{[x_0,k_n\cdot x_0]}$, where $k_n = g_1h_1...g_mh_m\hat{g}_n$ is a generating sequence of type (ii). Therefore $\gamma^{(ii)} \in \Lambda_{x_0}(g_1h_1...g_mh_mG\cdot x_0) \subseteq \bigcup\mathcal{Y}_G$, and thus $\bigcup\mathcal{Y}_G$ is dense in ∂X.

(v) Let $\gamma,\gamma' \in \partial_{x_0}X$ be any two geodesic rays such that γ,γ' do not belong to the same subset in $\mathcal{Y}$ and let $\gamma = \lim_{n\to\infty}\gamma_{[x_0,k_n\cdot x_0]}, \gamma' = \lim_{n\to\infty}\gamma_{[x_0,k'_n\cdot x_0]}$ for generating sequences k_n, k'_n which we can always assume dueto Lemma 6.7. Then k_n, k'_n are separated. Let k be any separator between k_n and k'_n. For simplicity of notation let K be the set of such generating sequences k''_n that there exists an $N \in \mathbb{N}$ such that for any natural $n > N$, the element k''_n is a prolongation of k. We will show that the set

$$Q = \left\{\gamma'' \in \partial_{x_0}X : \gamma'' = \lim_{n\to\infty}\gamma_{[x_0,k''_n\cdot x_0]},\ k''_n \in K\right\}$$

is clopen and $\mathcal{Y}$-saturated. Let $\gamma_1 \in Q$ and $\gamma_2 \in \partial_{x_0}X\backslash Q$. From Lemma 6.7 we know that $\gamma_1 = \lim_{n\to\infty}\gamma_{[x_0,k_n^{(1)}\cdot x_0]}$ and $\gamma_2 = \lim_{n\to\infty}\gamma_{[x_0,k_n^{(2)}\cdot x_0]}$, and from the definition of Q we know that the sequence $k_n^{(1)}$ for some point is a sequence of prolongations of k and no element of $k_n^{(2)}$ is a prolongation of k. Therefore from Lemma 6.10 we know that there exist $a \geqslant 1, b \geqslant 1$ such that $d_2\left(\gamma^{(1)},\gamma^{(2)}\right) > \frac{1}{a|k|_{\mathcal{A}}+b}$. Thus

$$\bigcup_{\gamma^{(1)}\in H} B\left(\gamma^{(1)}, \frac{1}{a\,|k|_{\mathcal{A}}} + b\right) = Q$$

$$\bigcup_{\gamma^{(2)}\in\partial_{x_0}\backslash Q} B\left(\gamma^{(2)}, \frac{1}{a\,|k|_{\mathcal{A}} + b}\right) = \partial_{x_0}X \setminus Q,$$

where B denotes a ball in Osajda's metric d_2. Therefore the set Q is clopen. Now let $Y \in \mathcal{Y}$ be any of the embedded copies of ΛG or ΛH. Then we know that Y is of form $\Lambda(\widetilde{k}G)$ or $\Lambda(\widetilde{k}H)$. Without loss of generality let us assume that $Y = \Lambda(\widetilde{k}G)$ and $\widetilde{k}$ has a normal form $g_1h_1...g_mh_m$. For every $\hat{\gamma},\widetilde{\gamma} \in \Lambda_{x_0}(\widetilde{k}G\cdot x_0)$

we know that $\hat{\gamma} = \lim_{n\to\infty} \gamma_{[x_0, g_1h_1...g_mh_m\hat{g}_n\cdot x_0]}$ and $\widetilde{\gamma} = \lim_{n\to\infty} \gamma_{[x_0, g_1h_1...g_mh_m\widetilde{g}_n\cdot x_0]}$, where $\hat{g}_n, \widetilde{g}_n \in G$. From the definition of prolongation we conclude that either both $g_1h_1...g_mh_m\hat{g}_n$ and $g_1h_1...g_mh_m\widetilde{g}_n$ are consist of prolongations of k or neither is. Therefore from the definition of Q either both $\hat{\gamma}, \widetilde{\gamma} \in Q$ or both $\hat{\gamma}, \widetilde{\gamma} \notin Q$ thus either $Y \subseteq Q$ or $Y \cap Q = \emptyset$. Therefore Q is $\mathcal{Y}$-saturated.

□

8 Open problems and concluding remarks

In this section we describe a few problems we encountered while working on the main theorem.

Let Γ be a group and let $\Gamma \curvearrowright X$ for a CAT(0) space X. For some subgroups G of Γ it appears that there is a non-empty, convex, closed, G-invariant subspace X_G of X such that $G \curvearrowright X_G$.

Example 8.1
For example if we take the group $\Gamma = \mathbb{Z}^2$ that acts on the CAT(0) space $X = \mathbb{R}^2$ by translations $(n, m)\cdot(x, y) = (x + n, y + m)$. If we take $G = \mathbb{Z} \times \{0\}$ subgroup of Γ then $\mathbb{R} \times \{0\}$ is a subspace of X having the properties described above.

Open Problem 8.2
Let $\Gamma = G * H$ be a group for some groups G, H and $\Gamma \curvearrowright X$ for a CAT(0) space X. Is there a non-empty, convex, closed subspace X_G of X such that $G \curvearrowright X_G$?

If the answer to the above question were to be affirmative, then it would be possible to reformulate the main theorem in terms of subspaces of X and their boundaries, because if such a subspace X_G would exist, then $\partial X_G = \Lambda G$. Personally I believe that if a subgroup is a factor in a free product, then a non-empty, convex, closed subspace does always exist.

Open Problem 8.3 is a more general approach to the observation from beginning of this section.

Open Problem 8.3
Let Γ be a group and X be a CAT(0) space such that $\Gamma \curvearrowright X$. What conditionsdoes a subgroup G of Γ need to meet for the existence of the subspace X_G?

It was shown in [2] (Proposition II.2.8) that if the subgroup G of Γ is finite, then there exists such a non-empty, convex, closed subspace of X. However, there are some groups that have infinite subgroups G for which the subspace X_G as described above cannot exist. It is not hard to observe that a necessary condition for existance of X_G is that the subgroup is undistorted in Γ. The following example of an infinite distorted subgroup in a CAT(0) group was presented in [7] (Theorem 1.6).

Example 8.4 (Distorted CAT(0) group)
Let $\Gamma = \langle a_0, a_1, a_2; a_0a_1 = a_1a_0a_2^{-1}a_0a_2 = a_1\rangle$. Then $\Gamma \curvearrowright X$ for a CAT(0) space X and Γ has a finitely generated free subgroup F_2 such that the distortion of F_2 is a polynomial function of degree 2.

Generalizing the main theorem of this thesis should also be possible. Firstly, by using the properties of dense amalgams described in [6], we can extend the definition of dense amalgam to the case where one or more of the X_i spaces are empty. This would allow us to describe the case of a free product of non-trivial groups. However, applying these changes requires some work to formally define all the cases and modify some of the lemmas.

Moreover, it should also be possible to generalize the main theorem to free products with amalgamation over a finite subgroup and HNN-extensions over finite subgroups. In the most general case, this can lead to a general theorem describing boundary at infinity for group graphs in which the edge groups are finite. This approach also requires modifying some lemmas, but in terms of the necessary ideas, it should not be significantly different from the methods presented in this thesis.

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