Harmonic analysis and Hardy spaces in the rational Dunkl setting

Summary of the dissertation

Dunkl theory is a generalization of Fourier analysis and special function theory related to root systems and reflection groups. The Dunkl operators T_j, which were introduced by Ch. Dunkl in 1989, are deformations of directional derivatives by difference operators related to a reflection group. They turn out to be the key tool in the study of special functions with reflection symmetries and allow one to built up a framework for a theory of special functions and integral transforms in several variables related to reflection groups. The aim of this thesis is to study harmonic analysis and Hardy spaces in this framework.

The first goal is to investigate, improve, and develop estimates for the Dunkl-type generalized translations of various functions (e.g. non-radial and sufficiently smooth) and express them in the spirit of analysis on spaces of homogeneous type, which is very useful from the point of view of applications to some harmonic analysis problems. In Chapter 3, we discuss the behavior of the heat kernel $h_t(\mathbf{x}, \mathbf{y})$ of the Dunkl heat semigroup generated by Dunkl–Laplace operator $\Delta_k = \sum_{j=1}^{N} T_j^2$. We provide the estimates in the spirit of analysis on spaces of homogeneous type with the additional factor $\left(1 + \frac{\|\mathbf{x}-\mathbf{y}\|^2}{t}\right)^{-1}$. Then we conclude the estimates of similar spirit for the Dunkl–Poisson kernel, generalized translations of radial compactly supported functions, and integral kernels associated with Dunkl-Bessel potentials.

In Chapter 4, we investigate various kinds of estimates for the generalized translations of not necessarily radial functions. First, we prove Theorem 4.1 concerning the supports of the generalized translations of L^2-functions. Using this theorem and some earlier estimates, we prove Theorem 4.8, which can be treated as a substitute for the L^1-boundedness of the generalized translations from the point of view of applications in harmonic analysis. Then, Theorem 4.8, together with the observation that any sufficiently regular function can be written as a convolution of a nice radial function ϕ with an L^1-function f, allows us to develop a method for estimating $\tau_{\mathbf{x}} g(-\mathbf{y})$ for many non-radial kernels g. This method turns out to be a key element in the proofs of many theorems in the further part of the dissertation in Chapters 5, 6, 7, 8, 9. It is done in the remaining part of Chapter 4.

In Part II of the dissertation we present some applications of the methods developed in Part I to a discussion of the boundedness of the Dunkl counterparts of some harmonic analysis operators. We consider:

- the weak type $(1, 1)$ and $L^p(dw)$-boundedness of maximal functions (Chapter 5);
- Hörmander's multiplier theorem for the Dunkl transform (Chapter 6);
- singular integrals of convolution type (Chapter 7);
- upper and lower bounds for the Littlewood–Paley square functions (Chapter 8);
- estimates of integral kernels of semigroups generated by sums of even powers of the Dunkl operators, for example $-\sum_{j=1}^{N}(-1)^{\ell} T_j^{2\ell}$ (Chapter 9).

Part III is devoted to the theory of the Hardy space $H^1_{\Delta_k}$ in the rational Dunkl setting. In Chapter 10, we prove that $H^1_{\Delta_k}$ admits characterizations by atomic decomposition with atoms in the sense of Coifmann–Weiss. As an application of the new atomic decompositions, we provide a version of Hörmander's multiplier theorem on $H^1_{\Delta_k}$. Then, in Chapter 11, we develop the theory of local Hardy spaces H^1_T, $T > 0$, in the context of Dunkl theory, which is parallel to the classical theory of Goldberg. We prove characterizations of them in terms of maximal functions, atomic decompositions, and relevant Riesz transforms.

Finally, in Part IV, we discuss the theory of the Dunkl–Schrödinger operators $L = -\Delta_k + V(\mathbf{x})$ with potentials $V(\mathbf{x})$ from the appropriate reverse Hölder classes. In Chapter 12 we elaborate the properties of auxiliary function m associated with V and prove the Fefferman–Phong inequality in the Dunkl setting. Then, we apply the results of Chapter 12 to study the behavior of eigenvalues of L (in Chapter 13) and the Hardy spaces associated with L (Chapter 14).

Wrocław, 22.02.2021

Agnieszka Hejna

Agnieszka Hejna