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Non-orientability of maps and combinatorics of Jack polynomials

ABSTRACT

In this Thesis we are interested in combinatorics of coefficients of the Jack polynomials, which are a certain family of symmetric polynomials. This combinatorics is related to enumeration of bicolored graphs.

Lassalle [Las08, Las09] initiated the investigation of the coefficient standing at $p_{\pi,1,1,\ldots}$ in the expansion of Jack symmetric polynomial $J_\lambda^{(\alpha)}$ in the basis of *power-sum symmetric functions*. With the right choice of the normalization this gives rise to quantities $\mathrm{Ch}_\pi^{(\alpha)}(\lambda)$ called *Jack characters*. One of the most interesting properties of Jack characters is that they can be expressed as polynomials in so called *free cumulants*, which are relatively simple functionals of the shape of the Young diagram λ. This expression is called *Kerov polynomial* and takes a particularly simple form [Bia03].

In this Thesis we study the relationship between combinatorics of coefficients of Kerov polynomials for Jack characters and enumeration of bicolored graphs. These graphs will have an additional structure, namely they will be *(non-oriented) maps*. Roughly speaking, such a map is a bicolored graph drawn on a (non-oriented, possibly non-orientable) surface. We give explicit formulas for coefficients of Kerov polynomials for Jack characters depending on summation over certain types of maps.

For a given map M, Dołęga, Féray and Śniady [DFŚ14] defined some polynomial $\mathrm{mon}_M(\gamma)$ in the indeterminate γ. This quantity is called *the measure of non-orientability* of M because, speaking very informally, it indeed measures how (non-)orientable the map M is. Presumably this quantity is the underlying structure behind the combinatorics of Jack polynomials and Jack characters.

Dołęga, Féray and Śniady [DFŚ14] defined the *orientability generating series* $\widehat{\mathrm{Ch}}_\pi^{(\alpha)}(\lambda)$

in the following way:

$$\widehat{\mathrm{Ch}}^{(\alpha)}_{\pi}(\lambda) := (-1)^{\ell(\pi)} \sum_{M} \mathrm{mon}_M \ \mathfrak{N}_M(\lambda), \tag{1}$$

where the sum runs over all non-oriented maps M with the face-type π and $\mathfrak{N}_M(\lambda)$ is the number of certain embeddings of a map M to the Young diagram λ. The content of this Thesis is a careful investigation of this quantity.

The main result of this thesis is the fact that if we restrict the set of maps to the maps with a specified small genus, Jack Characters coincide with the orientability generating series. We define $\widehat{\mathrm{Ch}}^{(\alpha),g}_{n}$ in the following way:

$$\widehat{\mathrm{Ch}}^{(\alpha),g}_{n}(\lambda) := (-1) \sum_{M} \mathrm{mon}_M \ \mathfrak{N}_M(\lambda), \tag{2}$$

where the sum runs over all non-oriented maps M with the face-type (n) and with genus g. The last expression coincides with the sum (1), except that we restrict the sum to the maps with a genus g.

With these definitions we investigate the following conjecture

Conjecture 1. *For genus $g \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}\right\}$ and arbitrary integer $n \geq 1$ the corresponding homogeneous parts of the Jack character and of the orientability generating series are equal:*

$$\mathrm{Ch}^{(\alpha),g}_{n} = \widehat{\mathrm{Ch}}^{(\alpha),g}_{n}.$$

We show that Conjecture 1. is true for genera 0, $\frac{1}{2}$ and 1. We also provide an explicit formula for Kerov polynomials for the homogeneous parts of the orientability generating series $\widehat{\mathrm{Ch}}^{(\alpha)}_{n}$ corresponding to genera 0, $\frac{1}{2}$, 1, $\frac{3}{2}$. The formula matches the form of the Kerov polynomials $\mathrm{Ch}^{(\alpha)}_{n}$ conjectured by Lassalle in [Las09, Conjecture 11.2, 11.3].

References

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